Highly compressible limit for Navier-Stokes equations

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1. Presentation of the results
2. Idea of the Proof

Let us recall the compressible Navier-Stokes equations:

- Mass equation :

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0,$$

- Momentum equation :

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div} u) + \varepsilon \nabla P(\rho) = 0,$$

- Initial data :

$$(\rho, u)_{/t=0} = (\rho_0, u_0).$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ the strain tensor.

We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $2\mu + N\lambda > 0$.

P is the pressure and we consider a pressure of the type $P(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma \geq 1$.

Remark

In the sequel we shall only consider the viscosity coefficients verifying the following equality (D. Bresch, B. Desjardins):

$$\lambda(\rho) = 2\rho\mu'(\rho) - 2\mu(\rho). \tag{1}$$

It includes the so called shallow water system.

Motivation of the choice on the viscosity coefficients

Proposition

Setting $\varphi'(\rho) = \frac{2\mu'(\rho)}{\rho}$ *and let* $P(\rho) = a\rho^\gamma$ *with* $\gamma > 1$. *Assume that* (ρ, u) *are classical solutions then for all* $t > 0$ *we have the two following entropy:*

$$\int_{\mathbb{R}^N} \frac{1}{2}\rho|u|^2(t,x) + \frac{a\varepsilon}{\gamma-1}\rho^\gamma(t,x)\,dx + \int_0^t\int_{\mathbb{R}^N} 2\mu(\rho)|Du|^2 dxdt + \int_0^t\int_{\mathbb{R}^N} \lambda(\rho)|\mathrm{div}u|^2 dxdt = \int_{\mathbb{R}^N} \rho_0|u_0|^2(x) + \frac{a\varepsilon}{\gamma-1}\rho_0^\gamma(x)\,dx. \tag{2}$$

$$\int_{\mathbb{R}^N} \frac{1}{2}\rho|u+\nabla\varphi(\rho)|^2(t,x)\,dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^N} \mu(\rho)|\nabla u -^t \nabla u|^2 dxdt + \int_0^t\int_{\mathbb{R}^N} \varepsilon\nabla P(\rho)\cdot\nabla\varphi(\rho) dxd = \int_{\mathbb{R}^N} \frac{1}{2}\rho_0|u_0+\nabla\varphi(\rho_0)|^2(x)\,dx. \tag{3}$$

and for any $\delta \in (0,2)$ *:*

$$\frac{d}{dt}\int_{\mathbb{R}^N} \rho\frac{1+|u|^2}{2}\ln(1+|u|^2)dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu(\rho)[1+\ln(1+|u|^2)|Du|^2 dx \leq C + C\varepsilon^2\Big(\int_{\mathbb{R}^N} \big(\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)}\big)^{\frac{2}{2-\delta}} dx\Big)^{\frac{2}{2-\delta}} \big(\int_{\mathbb{R}^N} (\rho|u|^2+\rho)dx\big)^{\frac{\delta}{2}}. \tag{4}$$

In order to obtain (2) we just multiply the momentum equation by u.

To get (3) we multiply the momentum equation by $\nabla\varphi(\rho)$ and we use integrations by parts. To obtain (4) it suffices to multiply the momentum equation by $(1+\ln(1+|u|^2))u$, we get:

$$\int \rho\frac{d}{dt}[\frac{1+|u|^2}{2}\ln(1+|u|^2)]dx + \int \rho u\cdot\nabla(\frac{1+|u|^2}{2}\ln(1+|u|^2))dx$$
$$+\nu_1\int \mu(\rho)(1+\ln(1+|u|^2)|D(u)|^2dx \leq \varepsilon|\int[1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma dx| \quad (5)$$
$$+C\int\mu(\rho)|\nabla u|^2dx.$$

It remains to bound the right hand side by integration by parts and Hölder's inequalities.

Entropy estimates

Roughly speaking we have the following estimate on ρ and u which are uniform in ε for all $T>0$:

- $\sqrt{\rho}u \in L^\infty_T(L^2)$,
- $\sqrt{\mu(\rho)}\nabla u$ and $\sqrt{\lambda(\rho)}\mathrm{div}u$ in $L^2_T(L^2)$,
- $\rho\frac{1+|u|^2}{2}\ln(1+|u|^2) \in L^\infty_T(L^1)$,
- $\rho \in L^\infty_T(L^1)$,
- $\rho|\nabla\varphi(\rho)|^2 \in L^\infty_T(L^1)$,

Remark

Since we aim at considering the highly compressible limit ($\varepsilon \to 0$) in the sense of the distribution it is particularly important to get uniform estimates in ε on (ρ, u). Indeed we need compactness arguments to pass to the limit.

Remark

Let us mention that the entropy (3) and (4) are not valid in the case of constant viscosity coefficients, in particular we have no uniform estimates on the density except the conservation of the mass.

Quasi solutions

Definition

We say that (ρ, u) is a quasi-solution of the compressible Navier-Stokes equation if (ρ, u) verify in distribution sense the pressureless system:

$$\begin{cases} \frac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \frac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)\,\mathrm{D}u) - \nabla(\lambda(\rho)\mathrm{div}u) = 0, \\ (\rho, u)_{\,t=0} = (\rho_0, u_0). \end{cases} \tag{6}$$

It is possible to reformulate the problem by introducing an effective velocity $v = u + \nabla\varphi(\rho)$ such that we have:

$$\begin{cases} \frac{\partial}{\partial t}\rho - \mathrm{div}(\rho\nabla\varphi(\rho)) + \mathrm{div}(\rho v) = 0, \\ \frac{\partial}{\partial t}(\rho v) + \mathrm{div}(\rho v \otimes u) - \mathrm{div}(\mu(\rho)\,\mathrm{curl}v) = 0, \\ (\rho, v)_{\,t=0} = (\rho_0, v_0). \end{cases} \tag{7}$$

Let us assume now to simplify that:

$$\mu(\rho) = \mu\rho^{\alpha} \;\text{ with } \alpha > 0 \;\text{ and }\; \lambda(\rho) = 2(\alpha - 1)\mu\rho^{\alpha}, \tag{8}$$

with $\alpha \geq 1 - \frac{1}{N}$ in order to ensure the relation $2\mu(\rho) + N\lambda(\rho) > 0$.

In this case we observe that it exists a quasi-solution $(\rho, -\nabla\varphi(\rho))$ when the initial data verify $u_0 = -\nabla\varphi(\rho_0)$ with ρ verifying the porous media or the fast diffusion equation when $\alpha \neq 1$:

$$\begin{cases} \partial_t \rho - 2\mu\Delta\rho^\alpha = 0, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{9}$$

Remark

The case $\alpha > 1$ (the porous media equations) arises as a model of slow diffusion of a gas inside a porous container. Unlike the heat equation $\alpha = 1$, this equation exhibits finite speed of propagation in the sense that solutions associated to compactly supported initial data remain compactly supported in space variable at all times). When $0 < \alpha < 1$, the opposite happens. Infinite speed of propagation occurs and solutions may even vanish in finite time. This problem is usually referred to as the fast diffusion equation.

Let us recall that there exists a theory of global unique solution for initial data $\rho_0 \in L^1(\mathbb{R}^N)$.

What happens in one dimension?

Let us consider the Navier-Stokes equations for compressible isentropic flows in one dimension:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0. \end{cases} \tag{10}$$

with possibly degenerate viscosity coefficient.
Setting $v = u + \partial_x \varphi(\rho)$ with $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$ we have:

$$\begin{cases} \partial_t \rho - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x \rho) + \partial_x(\rho v) = 0, \\ \rho\partial_t v + \rho u \partial_x v + \partial_x P(\rho) = 0. \end{cases} \tag{11}$$

Remark

The proof of global strong solution is proved by Mellet and Vasseur when $\mu(\rho) = \rho^\alpha$ *with* $0 < \alpha < \frac{1}{2}$.

Theorem (BH)

Assume that $\mu(\rho) = \mu\rho$ *and the initial data* ρ_0 *and* u_0 *satisfy:*

$$\begin{aligned} &0 < \alpha_0 \leq \rho_0(x) \leq \beta_0 < +\infty, \\ &\rho_0 - \bar{\rho} \in H^1(\mathbb{R}), \\ &u_0 - \bar{u} \in H^1(\mathbb{R}), \\ &\partial_x \rho_0 \in L^\infty, \end{aligned} \tag{12}$$

for some constant α_0 *and* β_0. *Assume that* $P(\rho) = \rho^\gamma$ *with* $\gamma \geq 2$. *Then there exists a global strong solution* (ρ, u) *of system (11) on* $\mathbb{R}^+ \times \mathbb{R}$ *such that for every* $T > 0$ *there exists constant* $\alpha(T)$ *and* $\beta(T)$ *such that*

$$0 < \alpha(T) \leq \rho(t,x) \leq \beta(T) < +\infty \;\; \forall (t,x) \in (0,T) \times \mathbb{R}.$$

Let us recall a classical result due to Solonnikov (76) of strong solution in finite time.

Proposition

Let (ρ_0, u_0) satisfy (12) and assume that μ satisfies $\mu(\rho) = \mu\rho^\alpha$ with $\alpha \geq 0$ then there exists T_0 depending on $\alpha_0, \beta_0, \|\rho_0 - \bar{\rho}\|_{H^1}$ and $\|u_0 - \bar{u}\|_{H^1}$ such that (10) has a unique solution (ρ, u) on $(0, T_0)$ satisfying:

$$\rho - \bar{\rho} \in L^\infty(0, T_1, H^1(\mathbb{R})),\ \partial_t \rho \in L^2((0, T_1) \times \mathbb{R}),$$
$$u - \bar{u} \in L^2(0, T_1, H^2(\mathbb{R})),\ \partial_t u \in L^2((0, T_1) \times \mathbb{R})$$

for all $T_1 < T_0$.
Moreover, there exist some $\alpha(T) > 0$ and $\beta(T) < +\infty$ such that $\alpha(t) \leq \rho(,x) \leq \beta(t)$ for all $t \in (0, T_0)$.

Proof of the the Theorem 2: In order to prove the theorem 1.1, it will be sufficient to show that we can control $\alpha(T)$, $\beta(T)$, $\|\rho(T) - \bar{\rho}\|_{H^1}$ and $\|u(T) - \bar{u}\|_{H^1}$ for any $T > 0$. In other words we are interested in proving that these quantities do not blow up when T goes to T_0.

It suffices to rewrite the momentum equation using the fact that:

$$\partial_x P(\rho) = \alpha\rho^\gamma \partial_x \ln\rho = c\rho^\gamma(v-u),$$

it gives then:

$$\rho\partial_t v + \rho u \partial_x v + c\rho^\gamma(v-u) = 0.$$

And it gives:

$$\partial_t v + u\partial_x v + c\rho^{\gamma-1} v = c\rho^{\gamma-1} u.$$

Next we show that ρu belongs in $L^2_T(L^\infty)$ using the fact that ρ is in L^∞ and $\sqrt{\rho}u$ and $\sqrt{\rho}\partial_x u$ belong respectively to $L^\infty(L^2)$ and $L^2(L^2)$. It is sufficient when $\gamma \geq 2$ to prove that v belongs in $L^\infty_T(L^\infty)$.

Using now the maximum principle on the first equation we prove that $\frac{1}{\rho}$ remains bound in $L^\infty_T(L^\infty)$.

It remains to bound $\|\rho(T)-\bar{\rho}\|_{H^1}$ and $\|u(T)-\bar{u}\|_{H^1}$ for any $T>0$ what is classical using the parabolic effects on u. ■

Let us recall some results concerning the existence of global weak and strong solution and the incompressible limit process.

Some results of global weak solution and of global strong solution

- D. Bresch and B. Desjardins [06], They proved new entropy provided that the viscosity coefficients verify relation (1). It implies that $\nabla\rho$ belongs in $L^\infty_T(L^2(\mathbb{R}^N))$.
- A. Mellet, A. Vasseur and C. Yu [07,15], Stability of the global weak solution with a gamma law, $P(\rho) = a\rho^\gamma$. The authors introduce a new entropy on the velocity which ensure a gain of integrability on u.
- BH [12], Existence of global strong solution with large initial data for the scaling of the equations. In particular we exhibit a family of large initial energy data in dimension $N = 2$ such that we have the existence of global strong solution.
- B. Desjardins, E. Grenier,P-L. Lions and N. Masmoudi [99], Results on the incompressible limit (which corresponds to take $\varepsilon = \frac{1}{\eta^2}$ with η the Mach number going to 0) in the framework of the global weak solutions for the ill-prepared data. One of the main ingredient of the proof is the use of Strichartz estimate.
- R. Danchin, L. He [01,14], Results on the incompressible limit for strong solutions in critical Besov spaces.

Remark

Let us mention that in the case of constant viscosity coefficients, we have the following results:

- *P-L Lions [98], Existence of global weak solution with large initial data in energy space* $\gamma > \frac{3}{2}$ *when* $N = 2$, $\gamma > \frac{9}{5}$ *when* $N = 3$, $\gamma > \frac{N}{2}$ *when* $N \geq 4$.
- *E. Feireisl, A. Novotny et al [02], existence of global weak solution when* $\gamma > \frac{N}{2}$.

Let us mention that in this case we can not obtain the entropy (3) and (4), in particular we do not get an uniform control on the density in terms of ε *when we consider the pressure* $P(\rho) = \varepsilon\rho^{\gamma}$.

It seems not clear how to pass to the limit when ε *goes to* 0. *Indeed we have not enough compactness information on* ρ *except that it is uniformly bounded in* ε *in* $L^{\infty}_{T}(L^1(\mathbb{R}^N))$.

On the porous medium equation

Let us mention also that the porous media equations are invariant by scaling, more precisely we can introduce a notion of self similarity $\rho(t,x) = t^{-\gamma} F(\frac{x}{t^{\beta}})$, with γ and β to be determined.

In our case γ and β have the form: $\gamma(\alpha-1)+2\beta=1$, and F verifies the following equation:

$$\Delta F^{\alpha} + \beta\eta\cdot\nabla F + \gamma F = 0.$$

It exists self similar solution such that the initial data is a Dirac mass, this is the so-called Barrenblatt solutions (or fundamental solution) that we can write under the following form:

$$U_m(t,x) = t^{-\gamma_1} F(\frac{x}{t^{\beta}}) \text{ with } F(x) = (C - \frac{(\alpha-1)\gamma_1}{2\alpha}|x|^2)_+^{\frac{1}{\alpha-1}}$$

with $C>0$ and $\gamma_1 = \frac{N}{N(\alpha-1)+2}$, $\beta = \frac{1}{N(\alpha-1)+2}$.

Remark

Here we have the conservation of the mass $\int U_m(t,x)dx = m$ with m depending on C and the initial data corresponds to the Dirac mass $m\delta_0$.

Similarly when $m_c < \alpha < 1$ with $m_c = \max(0, \frac{N-2}{N})$ it exists also Barrenblatt solutions defined as follows:

$$U_m(t,x) = t^{-\gamma_1} F(xt^{-\beta}) \text{ with } F(x) = (C + \kappa_1 |x|^2)_+^{\frac{-1}{\alpha-1}},$$

with $\kappa_1 = \frac{(1-\alpha)\gamma_1}{2N\alpha}$.

Remark

We recall that asymptotically in time all the global weak solution with L^1 initial data converges to a Barrenblatt solution determined by his mass $\|u_0\|_{L^1}$.

Remark

As we mentioned previously in the case of fast diffusion equation $0 < \alpha < 1$, infinite propagation occurs and solution may even vanish in finite time when α is in the interval $(0, m_c)$ with $m_c = \max(0, \frac{N-2}{N})$. In particular it implies a lost of the initial mass when ρ_0 is in L^1 (it implies also a lost of the regularity of the solution).

Theorem (BH)

Let $\mu(\rho) = \mu\rho^{\alpha}$ *with* $\alpha \geq 1 - \frac{1}{N}$. *Let* $\rho_0 \in L^1(\mathbb{R}^N)$ *with* $\rho_0 > 0$, *continuous and* $u_0 = -\nabla\varphi(\rho_0)$. *Then it exists a global weak quasi solution of the form* $(\rho, u = -\nabla\varphi(\rho))$ *with* (ρ, u) *belonging in* $C^{\infty}((0,+\infty)\times\mathbb{R}^N) \cap C([0,+\infty]\times\mathbb{R}^N)$ *and solving the following system almost everywhere :*

$$\partial_t \rho - 2\Delta\mu(\rho) = 0,\ \rho(0,\cdot) = \rho_0. \tag{13}$$

Furthermore we have:

$$\lim_{t\to+\infty} \|\rho(t) - U_m(t)\|_{L^1(\mathbb{R}^N)} = 0. \tag{14}$$

Convergence holds also in L^{∞} *norm:*

$$\lim_{t\to+\infty} t^{\beta}\|\rho(t) - U_m(t)\|_{L^{\infty}(\mathbb{R}^N)} = 0, \tag{15}$$

with $\beta = \frac{N}{N(\alpha-1)+2}$ *and* U_m *the Barrenblatt of mass* $m = \|\rho_0\|_{L^1(\mathbb{R}^N)}$. *Here the Barrenblatt solution are written under the following form:*

$$U_m(t,x) = t^{-\gamma_1} F(\frac{x}{t^{\beta}}) \ \textit{with} \ F(x) = (C - \frac{(\alpha-1)\gamma_1}{2\alpha}|x|^2)_+^{\frac{1}{\alpha-1}}$$

with $C > 0$ *and* $\gamma_1 = \frac{N}{N(\alpha-1)+2}$, $\beta = \frac{1}{N(\alpha-1)+2}$.

We obtain now a general result concerning the stability of the global weak quasi solution and a result of existence of global weak quasi solution for general initial data of the form $(\rho_0, -\nabla\varphi(\rho_0))$.

Theorem (BH)

Let $N \geq 2$. Assume that $\mu(\rho)$ and $\lambda(\rho)$ are two regular function of ρ. Let (ρ_n, u_n) be a sequence of global weak quasi solutions with initial data ρ_0^n and u_0^n such that:

$$\rho_0^n \geq 0, \;\; \rho_0^n \to \rho_0 \;\; in\; L^1(\mathbb{R}^N), \;\; \rho_0^n u_0^n \to \rho_0 u_0 \;\; in\; L^1(\mathbb{R}^N), \tag{16}$$

and satisfying the following bounds (with C constant independent on n):

$$\int_{\mathbb{R}^N} \rho_0^n \frac{|u_0^n|^2}{2} < C, \quad \int_{\mathbb{R}^N} \sqrt{\rho_0^n}|\nabla\varphi(\rho_0^n)|^2 dx < C \tag{17}$$

and:

$$\int_{\mathbb{R}^N} \rho_0^n \frac{1+|u_0^n|^2}{2} \ln(1+|u_0^n|^2)dx < C. \tag{18}$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n}u_n)$ converges strongly to a global weak quasi solution $(\rho, \sqrt{\rho}u)$ satisfying the energy inequalities (2), (3) and (4). If we assume moreover that (ρ_0, u_0) verify $u_0 = -\nabla\varphi(\rho)$ with $\mu(\rho) = \mu\rho^\alpha$ then it exists a global weak quasi solution (ρ, u) where ρ is solution of the porous media equation.

Theorem (BH)

Let $\gamma > 1$, $N \geq 2$. Assume that there exists a sequence of global weak solution $(\rho_\varepsilon, u_\varepsilon)$ of the compressible Navier-Stokes equations verifying our different entropies. Then $(\rho_\varepsilon, u_\varepsilon)$ converges in distribution sense to a global weak quasi-solution (ρ, u).

Furthermore the density ρ_ε converges strongly to ρ in $C([0,T], L^{1+\alpha}_{loc}(\mathbb{R}^N))$ with $0 < \alpha < \nu_1$ when $N \geq 2$;
$\sqrt{\rho_\varepsilon} u_\varepsilon$ converges strongly in $L^2(0,T,L^2_{loc})$ to $\sqrt{\rho} u$ and the momentum $m_\varepsilon = \rho_\varepsilon u_\varepsilon$ converges strongly to m in $L^1(0,T,L^1_{loc}(\mathbb{R}^N))$, for any $T > 0$.

Remark

When $u_0 = -\nabla\varphi(\rho_0)$ if the density ρ_ε of the highly compressible Navier-Stokes equation converges to ρ solution of the porous medium equation, we know that for a initial density with compact support the density ρ remains compactly support all along the time. Roughly speaking the previous result express the fact that the mass of $\rho_\varepsilon(t,\cdot)$ outside of the compact $supp \rho(t,\cdot)$ goes to 0 when ε goes to 0.

Theorem (BH, Ewelina Zatorska)

Let $N = 1$. Assume that $\gamma > 1$, $\alpha > 1$. Moreover assume that the initial data (ρ_0, u_0) satisfy $u_0 = -\partial_x \varphi(\rho_0)$. There exists a global weak solution $(\rho_\varepsilon, u_\varepsilon)$ in the sense of distribution. In addition, ρ_ε converges strongly to $\widetilde{\rho}$ the strong solution of the porous medium equation

$$\begin{cases} \partial_t \widetilde{\rho} - \frac{1}{\alpha} \partial_{xx} \widetilde{\rho}^\alpha = 0, \\ \widetilde{\rho}(0, x) = \rho_0(x), \end{cases} \tag{19}$$

in the following sense, there exists a constant $C > 0$ depending on ρ_0 such that

$$\|(\widetilde{\rho} - \rho_\varepsilon)(t)\|_{H^{-1}(\mathbb{R})} \leq C t^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}. \tag{20}$$

For $1 < \alpha \leq \frac{3}{2}$ there exists a constant $C > 0$ depending on ρ_0 such that

$$\|(\rho_\varepsilon - \widetilde{\rho})(t)\|_{L^2(\mathbb{R})} \leq C t^{\frac{1}{4}} \varepsilon^{\frac{1}{4}}$$

for all $t \geq 0$.Moreover assume that $supp \rho_0$ is compact and $1 \leq \alpha \leq \frac{3}{2}$ there exists a constant $C > 0$, $\alpha > 0$ depending only on ρ_0 such that for $\Omega_t = supp \widetilde{\rho}(t, \cdot)$:

$$\|\rho_\varepsilon(t) 1_{\Omega_t^c}\|_{L^1(\mathbb{R})} \leq C \varepsilon^{\frac{1}{4}} t^{\frac{1}{4}} (1 + t^{\frac{1}{2(\alpha+1)}}), \tag{21}$$

Idea of the Proof of the theorem 1.2:

Lemma

1. $\sqrt{\rho_n}$ *is bounded in* $L^\infty(0,T;H^1(\mathbb{R}^N))$
2. $\partial_t\sqrt{\rho_n}$ *is bounded in* $L^2(0,T;H^{-1}(\mathbb{R}^N))$.

Up to a subsequence $\sqrt{\rho_n}$ *converge almost everywhere and strongly to* ρ *in* $L^2(0,T;L^2_{loc}(\mathbb{R}^N))$.

$$\sqrt{\rho_n}\rightarrow\sqrt{\rho}\quad p.p\ et\ L^2_{loc}((0,T)\times\mathbb{R}^N)\ \text{strongly.}$$

Lemma

Up to a subsequence, the momentum $m_n=\rho_n u_n$ *converges strongly in* $L^{2-\varepsilon}(0,T;L^p_{loc}(\mathbb{R}^N))$ *(for any* $\varepsilon>0$ *small enough) to some* $m(x,t)$ *for all* $p\in[1,\frac{3}{2})$. *It implies that:*

$$\rho_n u_n\rightarrow m\quad\text{almost everywhere } (x,t)\in(0,T)\times\mathbb{R}^N.$$

Let us start with the case $N = 3$. We have:

$$\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n} u_n,$$

$\sqrt{\rho_n}$ is uniformly bounded in $L^\infty(L^6(\mathbb{R}^3))$ (because $\nabla\sqrt{\rho_n}$ is uniformly bounded in $L^\infty(L^2(\mathbb{R}^3))$) and $\sqrt{\rho_n}u_n$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^3))$ which implies that $\rho_n u_n$ is bounded in $L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$. Next we have:

$$\begin{aligned}\partial_i(\rho_n u_{nj}) &= \sqrt{\rho_n}\sqrt{\rho_n}\partial_i u_{nj} + \partial_i \rho_n u_{nj},\\ &= \sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj} + \frac{\partial_i \rho_n}{\sqrt{\rho_n}}\sqrt{\rho_n}u_{nj}.\end{aligned}$$

By entropy inequality (3) we know that $\frac{\partial_i \rho_n}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^3))$ which proves that $\frac{\partial_i \rho_n}{\sqrt{\rho_n}}\sqrt{\rho_n}u_{nj}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^3))$.

We proceed similarly for the term $\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj}$ which is uniformly bounded in $L^2_T(L^1(\mathbb{R}^N))$.

Hence for any compact K:

$$\nabla(\rho_n u_n) \text{ is bounded in } L^2(0,T;L^1(K))$$

In particular we have:

$$\rho_n u_n \text{ is bounded in } L^2(0,T;W^{1,1}(K))$$

It suffices now to use the Aubin Lions theorem, to do this we can observe that:

$$\partial_t(\rho_n u_n) \text{ is bounded in } L^2(0,T;W^{-2,\frac{4}{3}}(K)).$$

Lemma

$\sqrt{\rho_n}u_n$ strongly converge to $\sqrt{\rho}u$ in $L^2_{loc}((0,T)\times\mathbb{R}^N)$.

Let $M>0$ and K a compact:

$$\int_K |\sqrt{\rho_n}u_n-\sqrt{\rho}u|^2 dxdt \leq \int_K |\sqrt{\rho_n}u_n 1_{\{|u_n|\leq M\}}-\sqrt{\rho}u1_{\{|u|\leq M\}}|^2 dxdt$$
$$+2\int_K |\sqrt{\rho_n}u_n 1_{\{|u_n|\geq M\}}|^2 dxdt+2\int_K |\sqrt{\rho}u1_{\{|u|\geq M\}}|^2 dxdt,$$

By dominated convergence and Tchebytchev lemma for the first term and using the gain of integrability on $\sqrt{\rho_n}u_n$ for the second and third term allows to conclude to the strong L^2_{loc} convergence. □

Idea of the Proof of the theorem 3:
We now set

$$R_\varepsilon = \rho_\varepsilon - \widetilde{\rho},$$

which satisfies the following equation

$$\begin{cases} \partial_t R_\varepsilon - \dfrac{1}{\alpha}\partial_{xx}(\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha) + \partial_x(\rho_\varepsilon v_\varepsilon) = 0, \\ R_\varepsilon(0,x) = 0, \quad x \in \mathbb{R}, \end{cases}$$

at least in the sense of distributions.
Our goal is to estimate a relevant norm of R_ε in terms of ε using some ideas due to Vázquez. For a test function $\psi \in C_c^\infty((0,T] \times \mathbb{R})$, we obtain

$$\int_0^T \int_{\mathbb{R}} (R_\varepsilon \partial_t \psi + \frac{1}{\alpha}(\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha)\Delta\psi) + \int_0^T \int_{\mathbb{R}} (\rho_\varepsilon v_\varepsilon)\partial_x \psi - \int_{\mathbb{R}} (R_\varepsilon \psi)(T) = 0. \tag{23}$$

Let us now define

$$a(t,x) = \frac{\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha}{\rho_\varepsilon - \widetilde{\rho}} \tag{24}$$

when $\rho_\varepsilon \neq \widetilde{\rho}$ and $a = 0$ if $\rho_\varepsilon = \widetilde{\rho}$. This definition implies in particular that $\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha = aR_\varepsilon$. We can hence rewrite (23) as

$$\int_0^T \int_{\mathbb{R}} (R_\varepsilon(\partial_t \psi + \frac{1}{\alpha}a\Delta\psi) + \int_0^T \int_{\mathbb{R}} (\rho_\varepsilon v_\varepsilon)\partial_x \psi - \int_{\mathbb{R}} (R_\varepsilon \psi)(T) = 0. \tag{25}$$

The next step consists in solving the inverse problem in the interval $[-M, M] \subset \mathbb{R}$

$$\begin{cases} \partial_t \psi + \frac{1}{\alpha} a \Delta \psi = 0, \quad (t,x) \in [0,T] \times (-M,M), \\ \psi(t,-M) = \psi(t,M) = 0, \quad t \in [0,T], \\ \psi(T,x) = \theta(x), \quad x \in (-M,M), \end{cases} \tag{26}$$

where $\theta \in C_0^\infty((-M,M))$. Since a is not regular we are not in the classical theory of parabolic system, we need to approximate a by a regular sequence a_n verifying $\frac{1}{n} \leq a_n \leq n$.
We deduce finally that:

$$\begin{aligned} \Big| \int_{\mathbb{R}} R_\varepsilon(T)\theta \Big| &\leq T^{\frac{1}{2}} \|\partial_x \psi\|_{L^2_T(L^2(\mathbb{R})} \|\sqrt{\rho}_\varepsilon v_\varepsilon\|_{L^\infty_T(L^2(\mathbb{R})} \|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \\ &\leq C \|\partial_x \theta\|_{L^2(\mathbb{R})} \varepsilon^{\frac{1}{2}} T^{\frac{1}{2}}. \end{aligned} \tag{27}$$

It concludes the proof. □

Let us consider now an initial data with compact support. Using the maximum principle we deduce that $\text{supp}\widetilde{\rho}(t,\cdot)$ remains compact and is included in $[a_1 - C(t+t_1)^{\frac{1}{\alpha+1}}, b_1 + C(t+t_1)^{\frac{1}{\alpha+1}}]$ for some constants $-\infty < a_1 < b_1 < \infty$ and $C > 0$.
Now, we may deduce the following sequence of equalities and inequalities

$$\begin{aligned}
\|\rho_\varepsilon(t)1_{\Omega_t^c}\|_{L^1(\mathbb{R})} &\leq \|\rho_0\|_{L^1(\mathbb{R})} - \|\rho_\varepsilon(t)1_{\Omega_t}\|_{L^1(\mathbb{R})} \quad \text{(from the mass conservation)} \\
&\leq \|\widetilde{\rho}(t)1_{\Omega_t}\|_{L^1(\mathbb{R})} - \|\rho_\varepsilon(t)1_{\Omega_t}\|_{L^1(\mathbb{R})} \\
&\leq \|(\widetilde{\rho}(t) - \rho_\varepsilon(t))1_{\Omega_t}\|_{L^1(\mathbb{R})} \\
&\leq |\Omega_t|^{\frac{1}{2}} \|\widetilde{\rho}(t) - \rho_\varepsilon(t)\|_{L^2(\mathbb{R})} \quad \text{(from the Hölder inequality)} \\
&\leq C\varepsilon^{\frac{1}{4}} t^{\frac{1}{4}} (1 + t^{\frac{1}{2(\alpha+1)}}).
\end{aligned} \tag{28}$$

This finishes the proof of Theorem 3.