Numerical schemes for incompressible Stokes and Navier-Stokes problems

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2015 Porquerolles

3 chapters

1. Linear and nonlinear elliptic problems
2. Steady and transient Stokes problem
3. Steady and transient Navier-Stokes problem

3 numerical methods

1. conforming finite elements and Taylor-Hood scheme
2. nonconforming P^1 finite elements and Crouzeix-Raviart scheme
3. finite differences and MAC scheme

3 hours

Numerical approximation of linear elliptic problem

$$\begin{aligned} -\operatorname{div}(\Lambda\nabla\overline{u}) &= f \quad \text{in } \Omega \\ \overline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$\overline{u} \in H_0^1(\Omega)$ and $\forall \overline{v} \in H_0^1(\Omega)$

$$\int_\Omega \Lambda(\mathbf{x})\nabla\overline{u}(\mathbf{x}) \cdot \nabla\overline{v}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\overline{v}(\mathbf{x})\mathrm{d}\mathbf{x}$$

$$\underline{\lambda}|\xi|^2 \leq \xi \cdot \Lambda(\mathbf{x})\xi = \xi \cdot \Lambda(\mathbf{x})^T\xi \leq \overline{\lambda}|\xi|^2$$

approximated by

$$u \in X_{h,0},\ \forall v \in X_{h,0},\ \int_\Omega \Lambda(\mathbf{x})\nabla_h u(\mathbf{x}) \cdot \nabla_h v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_h v(\mathbf{x})\mathrm{d}\mathbf{x}$$

where

- $h \in]0, +\infty[$ index of the discretization among a family
- $X_{h,0} = \{(v_i)_{i\in I}, v_i \in \mathbb{R}\}$ vector space on $\mathbb{R}$ of the degrees of freedom
- $\Pi_h \ : \ X_{h,0} \to L^2(\Omega)$ reconstruction of function, linear operator
- $\nabla_h \ : \ X_{h,0} \to L^2(\Omega)^d$ reconstruction of gradient, linear operator
- such that $\|\nabla_h \cdot \|_{L^2(\Omega)^d}$ norm on $X_{h,0}$, denoted by $\| \cdot \|_h$

for any $i \in I$ let $v^{(i)} \in X_{h,0}$ defined by $v_i^{(i)} = 1$ and $v_j^{(i)} = 0$ for $i \neq j$

for $U = (u_i)_{i\in I}^T$: scheme equivalent to $AU = B$ with

$$a_{ij} = \int_\Omega \Lambda(\mathbf{x})\nabla_h v^{(i)}(\mathbf{x}) \cdot \nabla_h v^{(j)}(\mathbf{x})\mathrm{d}\mathbf{x} \text{ and } b_i = \int_\Omega f(\mathbf{x})\Pi_h v^{(i)}(\mathbf{x})\mathrm{d}\mathbf{x}$$

A invertible since $U^T AU = \int_\Omega \Lambda(\mathbf{x})\nabla_h u(\mathbf{x}) \cdot \nabla_h u(\mathbf{x})\mathrm{d}\mathbf{x}$ and $\|\nabla_h \cdot \|_{L^2(\Omega)^d}$ norm on $X_{h,0}$

if support $\nabla_h v^{(i)}$ small, matrix A is sparse

Conforming methods : $V_h = \{\Pi_h v, v \in X_{h,0}\} \subset H^1_0(\Omega)$ and $\nabla_h v = \nabla(\Pi_h v)$
Galerkin methods such as finite element methods, spectral methods

$$u \in X_{h,0},\ \forall v \in X_{h,0},\ \int_\Omega \Lambda(\mathbf{x})\nabla(\Pi_h u)(\mathbf{x}) \cdot \nabla(\Pi_h v)(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_h v(\mathbf{x})\mathrm{d}\mathbf{x}$$

Error estimate given by Céa's Lemma

$$\|\nabla\overline{u} - \nabla(\Pi_h u_h)\|_{L^2(\Omega)^d} \le \frac{\overline{\lambda}}{\underline{\lambda}} S_h(\overline{u})$$ and $$\|\overline{u} - \Pi_h u_h\|_{L^2(\Omega)} \le C_P \frac{\overline{\lambda}}{\underline{\lambda}} S_h(\overline{u})$$

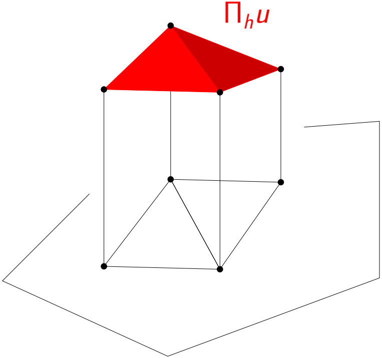
where

1. $$C_P = \max_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{\|v\|_{L^2(\Omega)}}{\|\nabla v\|_{L^2(\Omega)^d}}$$
2. $$S_h(\varphi) = \min_{v \in X_{h,0}} \left(\|\Pi_h v - \varphi\|^2_{L^2(\Omega)} + \|\nabla(\Pi_h v) - \nabla\varphi\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}},\ \forall \varphi \in H^1_0(\Omega)$$

family $\mathcal{D}$ such that, for all $h \in \mathcal{D} \subset]0, +\infty[$ with $0 \in \overline{\mathcal{D}}$

consistency $\forall \varphi \in H_0^1(\Omega),\ \lim_{h\to 0} S_h(\varphi) = 0$

Example : Conforming P^1 finite element method under regularity condition of the mesh



Need of P^2 finite elements for approximating the velocity in conforming methods

$$u \in X_{h,0},\ \forall v \in X_{h,0},\ \int_\Omega \Lambda(\mathbf{x})\nabla_h u(\mathbf{x}) \cdot \nabla_h v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_h v(\mathbf{x})\mathrm{d}\mathbf{x}$$

$\Pi_h v \in L^2(\Omega)$, $\nabla_h v \in L^2(\Omega)^d$

Error estimate given by Strang Lemma

$$\|\nabla\overline{u} - \nabla_h u_h\|_{L^2(\Omega)^d} \le \frac{1}{\underline{\lambda}}(W_h(\Lambda\nabla\overline{u}) + (\overline{\lambda} + \underline{\lambda})S_h(\overline{u}))$$

$$\|\overline{u} - \Pi_h u_h\|_{L^2(\Omega)} \le \frac{1}{\underline{\lambda}}(C_h W_h(\Lambda\nabla\overline{u}) + (C_h\overline{\lambda} + \underline{\lambda})S_h(\overline{u}))$$

where

1. $$C_h = \max_{v\in X_{h,0}\setminus\{0\}} \frac{\|\Pi_h v\|_{L^2(\Omega)}}{\|\nabla_h v\|_{L^2(\Omega)^d}}$$

2. $$S_h(\varphi) = \min_{v\in X_{h,0}} \left(\|\Pi_h v - \varphi\|^2_{L^2(\Omega)} + \|\nabla_h v - \nabla\varphi\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}},\ \forall\varphi \in H^1_0(\Omega)$$

3. $$W_h(\boldsymbol{\varphi}) = \max_{v\in X_{h,0}\setminus\{0\}} \frac{\int_\Omega (\nabla_h v(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x}) + \Pi_h v(\mathbf{x})\mathrm{div}\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x}}{\|\nabla_h v\|_{L^2(\Omega)^d}}$$
   $\forall\boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$

Remark : $\Lambda\nabla\overline{u} \in H_{\mathrm{div}}(\Omega)$

Sufficient condition for convergence of family of approximations

again family $\mathcal{D}$ such that, for all $h \in \mathcal{D} \subset]0, +\infty[$ with $0 \in \overline{\mathcal{D}}$

1. coercivity $\exists C_P \in \mathbb{R}$ s.t. $C_h \leq C_P$
2. consistency $\forall \varphi \in H_0^1(\Omega),\ \lim_{h\to 0} S_h(\varphi) = 0$
3. limit conformity $\forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega),\ \lim_{h\to 0} W_h(\boldsymbol{\varphi}) = 0$

Remark : regular functions are sufficient for checking consistency and limit conformity

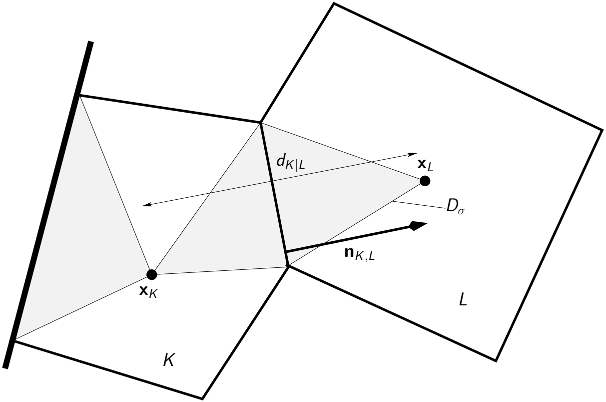
R dense in $H_0^1(\Omega)$ then $\lim_{h\to 0} S_h(\varphi) = 0,\ \forall \varphi \in \mathcal{R}$
equivalent to
$\lim_{h\to 0} S_h(u) = 0,\ \forall u \in H_0^1(\Omega)$

$\mathcal{S}$ dense in $H_{\mathrm{div}}(\Omega)$ and $C_h \leq C_P$ then $\lim_{h\to 0} W_h(\boldsymbol{\varphi}) = 0,\ \forall \boldsymbol{\varphi} \in \mathcal{S}$
equivalent to
$\lim_{h\to 0} W_h(\mathbf{U}) = 0,\ \forall \mathbf{U} \in H_{\mathrm{div}}(\Omega)$

$$\nabla_K u = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| u_\sigma \mathbf{n}_{K,\sigma}$$

Then $\displaystyle\int_\Omega \nabla_h u \cdot \boldsymbol{\varphi} = \sum_{\sigma = K|L \in \mathcal{F}} \frac{|\sigma| d_{K|L}}{d} u_\sigma \mathbf{n}_{K,L} \cdot d \frac{\boldsymbol{\varphi}_L - \boldsymbol{\varphi}_K}{d_{K|L}}$

and $d\mathbf{n}_{K,L} \cdot \dfrac{\boldsymbol{\varphi}_L - \boldsymbol{\varphi}_K}{d_{K|L}}$ defined on D_σ weakly convergent to $\mathrm{div}\boldsymbol{\varphi}$

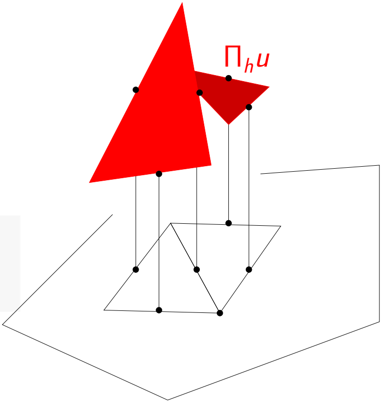


simplicial meshes, piecewise linear functions continuous at barycentre of faces $(\varphi_i)_{i=1,\dots,N}$

$$X_{h,0} = \mathbb{R}^N$$

$$\Pi_h u = \sum_{i=1}^N u_i \varphi_i$$

$$\nabla_h u = \sum_{i=1}^N u_i \nabla_T \varphi_i = \frac{1}{|T|} \sum_{\sigma \in \mathcal{F}_T} |\sigma| u_\sigma \mathbf{n}_{T,\sigma}$$



$X_{h,0}$ values in K

$\Pi_h u(\mathbf{x}) = u_K, \ \mathbf{x} \in K$

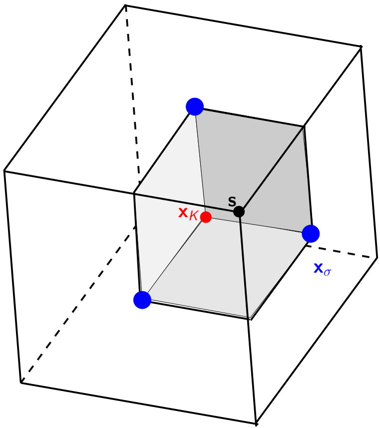
$$\nabla_{K,\mathbf{s}} u = \frac{1}{|K_\mathbf{s}|} \sum_{\sigma \in \mathcal{F}_{K,\mathbf{s}}} |\sigma| u_\sigma \mathbf{n}_{K_\mathbf{s},\sigma}$$

$$= \sum_{\sigma \in \mathcal{F}_{K,\mathbf{s}}} (u_\sigma - u_K) \frac{\mathbf{n}_{K,\sigma}}{d_{K,\sigma}}$$

if $\sigma = K|L$ then $u_\sigma = \dfrac{u_L d_{K,\sigma} + u_K d_{L,\sigma}}{d_{K,\sigma} + d_{L,\sigma}}$

if $\sigma \subset \partial\Omega$ then $u_\sigma = 0$

$$\nabla_h u(\mathbf{x}) = \nabla_{K,\mathbf{s}} u, \ \mathbf{x} \in V_{K,\mathbf{s}}$$



similar to finite difference/volume scheme
hence limit conformity, consistency
coercivity thanks to discrete functional analysis :

$$\|\Pi_h u\|^2_{L^2(\Omega)} \leq C_\Omega \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \frac{(u_\sigma - u_K)^2}{d_{K,\sigma}}$$

$$\begin{aligned} -\operatorname{div}(\Lambda(\overline{u})\nabla\overline{u}) &= f \quad \text{in } \Omega \\ \overline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$\overline{u} \in H_0^1(\Omega)$ and $\forall \overline{v} \in H_0^1(\Omega)$
$$\int_\Omega \Lambda(\overline{u}(\mathbf{x}),\mathbf{x})\nabla\overline{u}(\mathbf{x})\cdot\nabla\overline{v}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\overline{v}(\mathbf{x})\mathrm{d}\mathbf{x}$$
$\Lambda(s,\mathbf{x})$ continuous$/s$
$\underline{\lambda}|\xi|^2 \le \xi\cdot\Lambda(s,\mathbf{x})\xi = \xi\cdot\Lambda(s,\mathbf{x})^T\xi \le \overline{\lambda}|\xi|^2$

$$u_h \in X_{h,0},\ \forall v \in X_{h,0},\ \int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}),\mathbf{x})\nabla_h u_h(\mathbf{x})\cdot\nabla_h v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_h v(\mathbf{x})\mathrm{d}\mathbf{x}$$

where

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$\Pi_h\ :\ X_{h,0} \to L^2(\Omega)$ reconstruction of function, linear operator
$\nabla_h\ :\ X_{h,0} \to L^2(\Omega)^d$ reconstruction of gradient, linear operator
such that $\|\nabla_h\cdot\|_{L^2(\Omega)^d}$ norm on $X_{h,0}$, denoted by $\|\cdot\|_h$

need method to solve square system of nonlinear equations obtained letting $v = v^{(i)} \in X_{h,0}$ defined by $v_i^{(i)} = 1$ and $v_j^{(i)} = 0$ for $i \neq j$

Newton's method often efficient : $A^{(k)}(u_i^{(k+1)} - u_i^{(k)})_{i\in I}^T = B^{(k)}$

with $B_i^{(k)} = \int_\Omega f(\mathbf{x})\Pi_h v^{(i)}(\mathbf{x})\mathrm{d}\mathbf{x} - \int_\Omega \Lambda(\Pi_h u_h^{(k)}(\mathbf{x}),\mathbf{x})\nabla_h u_h^{(k)}(\mathbf{x})\cdot\nabla_h v^{(i)}(\mathbf{x})\mathrm{d}\mathbf{x}$

assume sufficient conditions on family of discretizations for convergence in the linear case

1. coercivity $\exists C_P \in \mathbb{R}$ s.t. $C_h \leq C_P$ with

$$C_h = \max_{v \in X_{h,0} \setminus \{0\}} \frac{\|\Pi_h v\|_{L^2(\Omega)}}{\|\nabla_h v\|_{L^2(\Omega)^d}}$$

2. consistency $\forall \varphi \in H_0^1(\Omega),\ \lim_{h \to 0} S_h(\varphi) = 0$ with

$$S_h(\varphi) = \min_{v \in X_{h,0}} \left(\|\Pi_h v - \varphi\|_{L^2(\Omega)}^2 + \|\nabla_h v - \nabla \varphi\|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}},\ \forall \varphi \in H_0^1(\Omega)$$

3. limit conformity $\forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega),\ \lim_{h \to 0} W_h(\boldsymbol{\varphi}) = 0$ with

$$W_h(\boldsymbol{\varphi}) = \max_{v \in X_{h,0} \setminus \{0\}} \frac{\int_\Omega \left(\nabla_h v(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) + \Pi_h v(\mathbf{x}) \mathrm{div} \boldsymbol{\varphi}(\mathbf{x})\right) \mathrm{d}\mathbf{x}}{\|\nabla_h v\|_{L^2(\Omega)^d}}$$
$$\forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$$

these properties are satisfies by P^1 conforming and non conforming, and by 7-point finite difference/volume schemes

convergence study under an additional hypothesis

use of coercivity $\|\Pi_h u_h\|_{L^2(\Omega)} \leq C_P \|\nabla_h u_h\|_{L^2(\Omega)^d}$
$v = u_h$ implies

$$\underline{\lambda} \|\nabla_h u_h\|^2_{L^2(\Omega)^d} \leq \int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}), \mathbf{x}) \nabla_h u_h(\mathbf{x}) \cdot \nabla_h u_h(\mathbf{x}) \mathrm{d}\mathbf{x}$$
$$= \int_\Omega f(\mathbf{x}) \Pi_h u_h(\mathbf{x}) \mathrm{d}\mathbf{x} \leq C_P \|f\|_{L^2(\Omega)} \|\nabla_h u_h\|_{L^2(\Omega)^d}$$

existence of a discrete solution by topological degree method

1. $F \ : \ V \times [0,1] \to \mathbb{R}$ continuous
2. $\exists C > 0, \ \forall (v, \theta) \in V \times [0,1], \ F(v, \theta) = 0 \Rightarrow \|v\| \leq C$
3. $F(\cdot, 0)$ affine

then $F(v, 1) = 0$ has at least one solution such that $\|v\| \leq C$

Application : $\Lambda_\theta(s, \mathbf{x}) = \theta \ \Lambda(s, \mathbf{x}) + (1 - \theta) \ \underline{\lambda}\mathrm{Id}$

Regularity of the limit

From estimate and coercivity, exists subsequence such that $\nabla_h u_h \rightharpoonup G$ in $L^2(\Omega)^d$ and $\Pi_h u_h \rightharpoonup \overline{u}$ in $L^2(\Omega)$
For any $\varphi \in H_{\mathrm{div}}(\Omega)$ **use of limit conformity** $W_h(\varphi) \to 0$

$$W_h(\varphi) \geq \frac{\left| \int_\Omega \left(\nabla_h u_h(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_h u_h(\mathbf{x}) \mathrm{div}\varphi(\mathbf{x})\right) \mathrm{d}\mathbf{x} \right|}{\|\nabla_h u_h\|_{L^2(\Omega)^d}} \to 0$$

implies (prolonge functions by 0 outside Ω)

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d)^d,\ \int_{\mathbb{R}^d} \left(G(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \overline{u}(\mathbf{x}) \mathrm{div}\varphi(\mathbf{x})\right) \mathrm{d}\mathbf{x} = 0$$

therefore $G = \nabla \overline{u}$ and $\overline{u} \in H_0^1(\Omega)$

for $\varphi \in H_0^1(\Omega)$, $I_h\varphi \in X_{h,0}$ such that

$$I_h\varphi = \mathrm{argmin}_{v \in X_{h,0}} \left(\|\Pi_h v - \varphi\|^2_{L^2(\Omega)} + \|\nabla_h v - \nabla\varphi\|^2_{L^2(\Omega)^d} \right)$$

use of consistency

$S_h(\varphi) \to 0$ implies $\Pi_h I_h \varphi \to \varphi$ in $L^2(\Omega)$ and $\nabla_h I_h\varphi \to \nabla\varphi$ in $L^2(\Omega)^d$

$$\int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}), \mathbf{x}) \nabla_h u_h(\mathbf{x}) \cdot \nabla_h I_h \varphi(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) \Pi_h I_h \varphi(\mathbf{x}) \mathrm{d}\mathbf{x}$$

If $\Lambda(\Pi_h u_h) = \Lambda_0$ pass to the limit by weak/strong convergence (get back the convergence result of the linear case)

How to pass to the limit if $\Lambda(\Pi_h u_h) \neq \Lambda_0$? weak convergence not sufficient assumption on the numerical method : **compactness**

$\|\nabla_h u_h\|_{L^2(\Omega)^d}$ bounded implies
existence subsequence such that $\Pi_h u_h$ converges in $L^2(\Omega)$ to some $\overline{u} \in L^2(\Omega)$

gives

$$\overline{u} \in H_0^1(\Omega) \text{ and } \forall \overline{v} \in H_0^1(\Omega), \int_\Omega \Lambda(\overline{u}(\mathbf{x}), \mathbf{x}) \nabla \overline{u}(\mathbf{x}) \cdot \nabla \overline{v}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) \overline{v}(\mathbf{x}) \mathrm{d}\mathbf{x}$$

Compactness hypothesis satisfied by conforming methods (Rellich theorem)
For the two nonconforming examples (nonconforming P^1 and 7-point finite difference) proof by discrete functional analysis

$$\|\Pi_h u(.+\xi) - \Pi_h u\|_{L^1(\Omega)} \leq C_\Omega |\xi| \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \frac{(u_\sigma - u_K)^2}{d_{K,\sigma}} \right)^{1/2}$$

and discrete Sobolev inequality, for $p \leq 2d/(d-2)$

$$\|\Pi_h u\|_{L^p(\Omega)} \leq C \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \frac{(u_\sigma - u_K)^2}{d_{K,\sigma}} \right)^{1/2}$$

Passing to the limit on

$$\int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}),\mathbf{x})\nabla_h u_h(\mathbf{x})\cdot\nabla_h u_h(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\Pi_h u_h(\mathbf{x})\mathrm{d}\mathbf{x}$$

implies (using $\overline{u}$ both weak solution of continuous problem and test function)

$$\lim_{h\to 0}\int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}),\mathbf{x})\nabla_h u_h(\mathbf{x})\cdot\nabla_h u_h(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})\overline{u}(\mathbf{x})\mathrm{d}\mathbf{x}$$
$$= \int_\Omega \Lambda(\overline{u}(\mathbf{x}),\mathbf{x})\nabla\overline{u}(\mathbf{x})\cdot\nabla\overline{u}(\mathbf{x})\mathrm{d}\mathbf{x}$$

using $\nabla_h u_h \rightharpoonup \nabla\overline{u}$ in $L^2(\Omega)^d$ we get

$$\lim_{h\to 0}\int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}),\mathbf{x})(\nabla_h u_h(\mathbf{x})-\nabla\overline{u}(\mathbf{x}))\cdot(\nabla_h u_h(\mathbf{x})-\nabla\overline{u}(\mathbf{x}))\mathrm{d}\mathbf{x} = 0$$

which proves strong convergence of gradient since

$$\underline{\lambda}\|\nabla_h u_h - \nabla\overline{u}\|_{L^2(\Omega)^d} \le \int_\Omega \Lambda(\Pi_h u_h(\mathbf{x}),\mathbf{x})(\nabla_h u_h(\mathbf{x})-\nabla\overline{u}(\mathbf{x}))\cdot(\nabla_h u_h(\mathbf{x})-\nabla\overline{u}(\mathbf{x}))\mathrm{d}\mathbf{x}$$

$$\begin{aligned} \eta\overline{u} - \Delta\overline{u} + \nabla\overline{p} &= f & \text{in } \Omega \\ \mathrm{div}\overline{u} &= 0 & \text{in } \Omega \\ \overline{u} &= 0 & \text{on } \partial\Omega \\ \int_\Omega \overline{p}(\mathbf{x})\mathrm{d}\mathbf{x} &= 0 \end{aligned}$$

$$\begin{aligned} &(\overline{u},\overline{p}) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \\ &\eta \int_\Omega \overline{u}\cdot\overline{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla\overline{u} : \nabla\overline{v}\,\mathrm{d}\mathbf{x} - \int_\Omega \overline{p}\,\mathrm{div}\overline{v}\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega f\cdot\overline{v}\,\mathrm{d}\mathbf{x},\ \forall\overline{v} \in \mathbf{H}^1_0(\Omega) \\ &\mathrm{div}\overline{u} = 0 \text{ a.e. in } \Omega \end{aligned}$$

Numerical scheme

$$\begin{aligned} &(u,p) \in X_{h,0} \times Y_{h,0},\ \forall v \in X_{h,0} \\ &\eta \int_\Omega \Pi_h u \cdot \Pi_h v\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla_h u : \nabla_h v\,\mathrm{d}\mathbf{x} - \int_\Omega \chi_h p\,\mathrm{div}_h v\,\mathrm{d}\mathbf{x} = \int_\Omega f\cdot\Pi_h v\,\mathrm{d}\mathbf{x} \\ &\forall q \in Y_{h,0},\ \int_\Omega \mathrm{div}_h u\ \chi_h q\,\mathrm{d}\mathbf{x} = 0 \end{aligned}$$

where

$X_{h,0} = \{(v_i)_{i\in I}, v_i \in \mathbb{R}\}$ vector space on $\mathbb{R}$ of the degrees of freedom of the velocity
$\Pi_h\ :\ X_{h,0} \to \mathbf{L}^2(\Omega)$ reconstruction of velocity, linear operator
$\nabla_h\ :\ X_{h,0} \to \mathbf{L}^2(\Omega)^d$ reconstruction of gradient, linear operator
such that $\|\nabla_h \cdot\|_{\mathbf{L}^2(\Omega)^d}$ norm on $X_{h,0}$, denoted by $\|\cdot\|_h$
$\mathrm{div}_h\ :\ X_{h,0} \to L^2(\Omega)$ reconstruction of divergence, linear operator

$Y_h = \{(p_j)_{j\in J}, p_j \in \mathbb{R}\}$ vector space on $\mathbb{R}$ of the degrees of freedom of the pressure
$\chi_h\ :\ Y_h \to L^2(\Omega)$ reconstruction of pressure, linear operator,
such that $\|\chi_h \cdot\|_{L^2(\Omega)}$ norm on Y_h and $Y_{h,0} = \{p \in Y_h, \int_\Omega \chi_h p(\mathbf{x})\mathrm{d}\mathbf{x} = 0\}$

Error estimate

$$\|\nabla \overline{u} - \nabla_h u\|_{L^2} + \|\overline{p} - \chi_h p\|_{L^2} \leq \frac{a + bC_h}{\beta_h}\left(\overline{W}_h(\nabla \overline{u}, \overline{p}) + S_h(\overline{u}) + \widetilde{S}_h(\overline{p})\right)$$ where

1. $$C_h = \max_{v \in X^*_{h,0}} \left(\frac{\|\Pi_h v\|_{L^2(\Omega)}}{\|\nabla_h v\|_{\mathbf{L}^2(\Omega)^d}} \right) + \max_{v \in X^*_{h,0}} \left(\frac{\|\mathrm{div}_h v\|_{L^2(\Omega)}}{\|\nabla_h v\|_{\mathbf{L}^2(\Omega)^d}} \right)$$
2. $$\beta_h = \min_{q \in Y^*_{h,0}} \left(\max_{v \in X^*_{h,0}} \left(\frac{\int_\Omega \chi_h q \, \mathrm{div}_h v \, \mathrm{d}x}{\|\nabla_h v\|_{\mathbf{L}^2(\Omega)^d} \|\chi_h q\|_{L^2(\Omega)}} \right) \right)$$ assumed to be > 0
3. $S_h : \mathbf{H}^1_0(\Omega) \to [0, +\infty)$

   $$\varphi \mapsto \min_{v \in X_{h,0}} \left(\|\Pi_h v - \varphi\|_{\mathbf{L}^2(\Omega)} + \|\nabla_h v - \nabla \varphi\|_{\mathbf{L}^2(\Omega)^d} + \|\mathrm{div}_h v - \mathrm{div} \varphi\|_{L^2(\Omega)} \right)$$

   $$\widetilde{S}_h \ : \ L^2_0(\Omega) \to [0, +\infty), \ \psi \mapsto \min_{z \in Y_{h,0}} \|\chi_h z - \psi\|_{L^2(\Omega)}$$
4. $\overline{W}_h : Z(\Omega) := \{(\boldsymbol{\varphi}, \psi) \in \mathbf{L}^2(\Omega)^d \times L^2(\Omega), \mathrm{div} \ \boldsymbol{\varphi} - \nabla \psi \in \mathbf{L}^2(\Omega)\} \mapsto [0, +\infty)$

   $$(\boldsymbol{\varphi}, \psi) \mapsto \max_{v \in X^*_{h,0}} \left(\frac{\int_\Omega [\nabla_h v : \boldsymbol{\varphi} + \Pi_h v \cdot (\mathrm{div} \ \boldsymbol{\varphi} - \nabla \psi) - \psi \mathrm{div}_h v] \ \mathrm{d}x}{\|\nabla_h v\|_{\mathbf{L}^2(\Omega)^d}} \right)$$

1/ $(\nabla \overline{u}, \overline{p}) \in Z(\Omega)$
2/ conforming method :
$\Pi_h u \in \mathbf{H}^1_0(\Omega)$, $\nabla_h u = \nabla(\Pi_h u)$, $\mathrm{div}_h u = \mathrm{div}(\Pi_h u)$, $\overline{W}_h(\boldsymbol{\varphi}, \psi) = 0$

for any $i \in I$ let $v^{(i)} \in X_{h,0}$ defined by $v_i^{(i)} = 1$ and $v_{i'}^{(i)} = 0$ for $i' \neq i$

for any $j \in J^*$ let $p^{(j)} \in Y_{h,0}$ be a basis function such that $\int_\Omega \chi_h p^{(j)}(\mathbf{x})\mathrm{d}\mathbf{x} = 0$

Then for $U = (u_i)_{i\in I}^T$, $P = (p_j)_{j\in J^*}^T$: scheme equivalent to

$$\begin{pmatrix} A & D \\ D^T & 0 \end{pmatrix} \begin{pmatrix} U \\ -P \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$ with

$$a_{ij} = \eta \int_\Omega \Pi_h v^{(i)}(\mathbf{x}) \cdot \Pi_h v^{(j)}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_\Omega \nabla_h v^{(i)}(\mathbf{x}) : \nabla_h v^{(j)}(\mathbf{x})\mathrm{d}\mathbf{x}$$

$$d_{ij} = \int_\Omega \mathrm{div}_h v^{(i)}(\mathbf{x})\ \chi_h p^{(j)}(\mathbf{x})\ \mathrm{d}\mathbf{x} \text{ and } b_i = \int_\Omega f(\mathbf{x}) \cdot \Pi_h v^{(i)}(\mathbf{x})\mathrm{d}\mathbf{x}$$

implementation may be different but algebraically equivalent

from estimate with $\overline{u} = 0$ and $\overline{p} = 0$ sufficient condition for existence and uniqueness of solution

$\|\nabla_h \cdot\|_{\mathbf{L}^2(\Omega)^d}$ norm on $X_{h,0}$, denoted by $\|\cdot\|_h$

$\|\chi_h \cdot\|_{L^2(\Omega)}$ norm on Y_h

$$\beta_h = \min_{q\in Y_{h,0}^*} \left(\max_{v\in X_{h,0}^*} \left(\frac{\int_\Omega \chi_h q\ \mathrm{div}_h v\ \mathrm{d}x}{\|\nabla_h v\|_{\mathbf{L}^2(\Omega)^d} \|\chi_h q\|_{L^2(\Omega)}} \right) \right) > 0$$

note : symmetric system, non positive, solved by iterative methods note that since system not SDP, no a priori convergence of conjugate gradient method

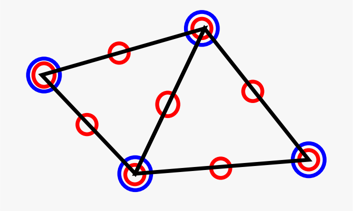
elliptic problem for each of the components of the velocity
pressure correction method, SIMPLE algorithm
use of transient approximation for steady problem

1. coercivity : C_h bounded by above, β_h bounded by below
2. consistency : $\forall \varphi \in \mathbf{H}_0^1(\Omega)$, $S_h(\varphi) \to 0$, $\forall \psi \in L^2(\Omega)$, $\widetilde{S}_h(\psi) \to 0$
3. limit-conformity : $\forall (\boldsymbol{\varphi}, \psi) \in Z(\Omega)$, $\overline{W}_h(\boldsymbol{\varphi}, \psi) \to 0$

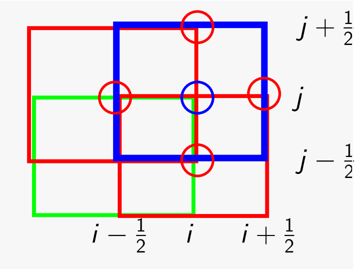
Hold in the 3 examples

1. Taylor-Hood
2. Crouzeix-Raviart
3. MAC

Practical proof that β_h bounded by below : use Necas result
exists $\overline{v} \in \mathbf{H}_0^1(\Omega)$ such that $\mathrm{div}\overline{v} = \chi_h p$ and $\|\nabla \overline{v}\|_{\mathbf{L}^2} \leq C_\Omega \|\chi_h p\|_{L^2}$
then interpolate $\overline{v}$ in $X_{h,0}$



1. $X_{h,0}$: vector space of the degrees of freedom $(P_2)^2$ ◯
2. Y_h : vector space of the degrees of freedom P_1 ◯
3. $\Pi_h u = \sum_{i \in I} u_i \varphi_i$, where φ_i is P_2 basis function
4. $\chi_h p = \sum_{j \in J} p_j \psi_j$, where ψ_j is P_1 basis function
5. $\nabla_h u = \nabla(\Pi_h u)$
6. $\mathrm{div}_h u = \mathrm{div}(\Pi_h u)$

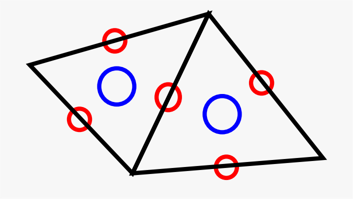


1. $X_{h,0} = \{(u_{i+\frac{1}{2},j}, u_{i,j+\frac{1}{2}})_{i,j}\}$ values at edges of blue mesh
2. $Y_h = \{(p_{i,j})_{i,j}\}$ values at center of blue mesh
3. $\Pi_h u$ constant in red staggered meshes
4. $\chi_h p$ constant in blue mesh
5. $\nabla_h^{(1,1)} u = \dfrac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}$ in blue mesh

   $\nabla_h^{(1,2)} u = \dfrac{u_{i-\frac{1}{2},j} - u_{i-\frac{1}{2},j-1}}{y_j - y_{j-1}}$ in green mesh

   $\nabla_h^{(2,1)} u = \dfrac{u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{x_i - x_{i-1}}$ in green mesh

   $\nabla_h^{(2,2)} u = \dfrac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}}$ in blue mesh
6. $\mathrm{div}_h u = \nabla_h^{(1,1)} u + \nabla_h^{(2,2)} u$ in blue mesh



1. $X_{h,0}$: vector space of all families of vectors at ◯
2. Y_h : vector space of all families of values at ◯
3. Π_h : nonconforming piecewise affine reconstruction of u
4. χ_h : piecewise constant reconstruction of p
5. ∇_h : so-called "broken gradient"
6. div_h : piecewise constant value at ◯ obtained through the balance of the normal velocities
   again satisfies $\mathrm{div}_h u = \nabla_h^{(1,1)} u + \nabla_h^{(2,2)} u$

Weak formulation :

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + \nabla \overline{p} & = f & \text{in } \Omega \times (0,T) \\ \operatorname{div} \overline{u} & = 0 & \text{in } \Omega \times (0,T) \\ \overline{u} & = 0 & \text{on } \partial\Omega \times (0,T) \\ \overline{u}(\cdot, 0) & = \overline{u}_{\text{ini}} & \text{a.e on } \Omega, \end{cases}$$

weak formulation

$$\begin{aligned} & \overline{u} \in L^2(0,T; E(\Omega)) \cap C([0,T]; \mathbf{L}^2(\Omega)),\ \partial_t \overline{u} \in L^2(0,T; \mathbf{H}^{-1}(\Omega)) \\ & \overline{p} \in L^2(0,T; L^2_0(\Omega)) \\ & \int_0^T \int_\Omega \langle \partial_t \overline{u}; \varphi \rangle \, dxdt + \int_0^T \int_\Omega \nabla \overline{u} : \nabla \varphi \, dxdt - \int_0^T \int_\Omega \overline{p}(x,t) \operatorname{div} \varphi \, dxdt \\ & \qquad = \int_0^T \int_\Omega f \cdot \varphi \, dxdt,\ \forall \varphi \in L^2(0,T; \mathbf{H}^1_0(\Omega)) \\ & \overline{u}(\cdot, 0) = \overline{u}_{\text{ini}} \text{ a.e on } \Omega \end{aligned}$$

where $E(\Omega) = \{v \in \mathbf{H}^1_0(\Omega),\ \operatorname{div}(v) = 0\}$

$$u_h^{(0)} = J_h u_{\mathrm{ini}}, \quad u_h^{(n+1)} \in X_{h,0}\,, \ p_h^{(n+1)} \in Y_{h,0}\,,$$

$$\int_\Omega \Pi_h \delta_h^{n+\frac{1}{2}} u_h \cdot \Pi_h v \; dx + \int_\Omega \nabla_h u_h^{(n+1)} : \nabla_h v \; dx - \int_\Omega \chi_h p_h^{(n+1)} \mathrm{div}_h v \; dx$$

$$= \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h v \; dxdt, \quad \forall v \in X_{h,0}$$

$$\int_\Omega \mathrm{div}_h u_h^{(n+1)} \chi_h q \; dx = 0, \quad \forall q \in Y_{h,0} \qquad \text{and } \delta_h^{n+\frac{1}{2}} u_h = \frac{u_h^{(n+1)} - u_h^{(n)}}{\delta t^{n+\frac{1}{2}}}$$

- $(X_{h,0}, \Pi_h, \nabla_h, Y_h, \chi_h, \mathrm{div}_h)$ defined as in the steady case
- $J_h : \mathbf{L}^2(\Omega) \mapsto X_{h,0}$ interpolation operator
- $t^{(0)} = 0 < t^{(1)} < ... < t^{(N)} = T$, $\delta t^{n+\frac{1}{2}} = t^{(n+1)} - t^{(n)}$ for all $n = 0, ..., N-1$ and $\delta t_h = \max_{n=0,\ldots,N-1} (\delta t^{n+\frac{1}{2}})$

Assumptions for the convergence study

1. coercivity : C_h bounded by above, β_h bounded by below
2. consistency : $\forall \varphi \in \mathbf{H}_0^1(\Omega)$, $S_h(\varphi) \to 0$, $\forall \psi \in L^2(\Omega)$, $\widetilde{S}_h(\psi) \to 0$
3. limit-conformity : $\forall (\boldsymbol{\varphi}, \psi) \in Z(\Omega)$, $\overline{W}_h(\boldsymbol{\varphi}, \psi) \to 0$
4. for all $\varphi \in \mathbf{L}^2(\Omega)$, $\Pi_h J_h \varphi \to \varphi$ in $\mathbf{L}^2(\Omega)$
5. $\delta t_h \to 0$ as $h \to 0$

estimate on the velocity

$v = \delta t^{n+\frac{1}{2}} u_h^{(n+1)}$ and $q = p_h^{(n+1)}$ in scheme leads to

$$\int_\Omega \left(\Pi_h u_h^{(n+1)} - \Pi_h u_h^{(n)} \right) \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_h u_h^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t = \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x\mathrm{d}t$$

$(a - b) \cdot a \geq \frac{1}{2}(a^2 - b^2)$ implies

$$\frac{1}{2} \int_\Omega \left[\left| \Pi_h u_h^{(n+1)} \right|^2 - \left| \Pi_h u_h^{(n)} \right|^2 \right] \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_h u_h^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x\mathrm{d}t$$

so $\frac{1}{2} \|\Pi_h u_h^{(m)}\|^2_{\mathbf{L}^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_h u_h\|^2_{\mathbf{L}^2(\Omega)^d} \mathrm{d}t \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_h u_h \mathrm{d}x\mathrm{d}t + \frac{1}{2} \|\Pi_h J_h u_{\mathrm{ini}}\|^2_{\mathbf{L}^2(\Omega)}$

provides, using coercivity, consistency of initial interpolation and Young inequality

$\Pi_h u_h$ bounded in $L^\infty(0, T; \mathbf{L}^2(\Omega))$

$\nabla_h u_h$ bounded in $\mathbf{L}^2(\Omega \times (0, T))^d$

Regularity of the limit

$\Pi_h u_h \rightharpoonup \overline{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ in $L^\infty(0,T;\mathbf{L}^2(\Omega))$
$\nabla_h u_h \rightharpoonup \zeta$ in $\mathbf{L}^2(\Omega \times (0,T))^d$
$\varphi \in C^\infty(\mathbb{R}^d)^d$, $\theta \in C_c^\infty(]0,T[)$, $\int_0^T \int_\Omega (\nabla_h u_h : \varphi + \Pi_h u_h \cdot \mathrm{div}\varphi)\theta \mathrm{d}x\mathrm{d}t \leq C\overline{W}_h(\varphi, 0)$

implies $$\int_0^T \int_{\mathbb{R}^d} \zeta : (\varphi\,\theta) + \overline{u} \cdot \mathrm{div}(\varphi\,\theta)\mathrm{d}x\mathrm{d}t = 0$$

and therefore $\overline{u} \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ and $\zeta = \nabla\overline{u}$

$\psi \in H^1(\Omega)$, $\int_\Omega \psi = 0$, $\int_0^T \int_\Omega \Pi_h u \cdot \nabla\psi\,\theta + \mathrm{div}_h u_h\,\psi\,\theta \mathrm{d}x\mathrm{d}t \leq C\overline{W}_h(0,\psi)$
$$\int_0^T \int_\Omega \Pi_h u \cdot \nabla\psi\,\theta \mathrm{d}x\mathrm{d}t \leq C_\theta ||\mathrm{div}_h u_h||_{L^2} \;\; ||\chi_h \widetilde{I}_h \psi - \psi||_{L_0^2} + C\overline{W}_h(0,\psi)$$

implies $$\int_0^T \int_\Omega \overline{u} \cdot \nabla\psi\theta \mathrm{d}x\mathrm{d}t = 0$$ and therefore $\mathrm{div}\overline{u} = 0$ a.e. in $\Omega\times]0,T[$

$\theta \in C_c^\infty([0,T))$ $w \in E(\Omega)$ then

$$(w_h, r) \in X_{h,0} \times Y_{h,0},\ \forall v \in X_{h,0}$$
$$\int_\Omega \nabla_h w_h : \nabla_h v \,\mathrm{d}\mathbf{x} - \int_\Omega \chi_h r \,\mathrm{div}_h v \,\mathrm{d}\mathbf{x} = \int_\Omega \nabla w : \nabla_h v \,\mathrm{d}\mathbf{x}$$
$$\forall q \in Y_{h,0},\ \int_\Omega \mathrm{div}_h w_h \ \chi_h q \,\mathrm{d}\mathbf{x} = 0$$

then $\Pi_h w_h \to w$ and $\nabla_h w_h \to \nabla w$ in L^2

$T_1 + T_2 + T_3 = T_4$ with

$$T_1 = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \Pi_h \delta^{(n+\frac{1}{2})} u_h \cdot \Pi_h w_h \mathrm{d}x \qquad T_2 = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \nabla_h u_h^{(n+1)} : \nabla_h w_h \mathrm{d}x$$

$$T_3 = -\sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \chi_h p_h^{(n+1)} \mathrm{div}_h w_h \mathrm{d}x \qquad T_4 = \sum_{n=0}^{N-1} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h w_h \mathrm{d}x\mathrm{d}t$$

$$T_1 = -\int_0^T \int_\Omega \theta' \Pi_h u_h \cdot \Pi_h w_h \mathrm{d}x\mathrm{d}t - \theta(0) \int_\Omega \Pi_h u_h^{(0)} \cdot \Pi_h w_h \mathrm{d}x.$$

$$T_1 \to -\int_0^T \int_\Omega \theta' \overline{u} \cdot w \mathrm{d}x\mathrm{d}t - \theta(0) \int_\Omega \overline{u}_{\mathrm{ini}} \cdot w \mathrm{d}x.$$

$$T_2 \to \int_0^T \theta \int_\Omega \nabla \overline{u} : \nabla w \mathrm{d}x\mathrm{d}t \quad \text{and} \quad T_4 \to \int_0^T \theta \int_\Omega f \cdot w \mathrm{d}x\mathrm{d}t$$

Then strong convergence of the function and of its gradient from convergence of the norms and energy estimate

$G = 0$, $u_{\rm ini} \in E(\Omega)$, $(||J_h u_{\rm ini}||_h)_{h\in\mathcal{D}}$ bounded

for all $h \in \mathcal{D}$, $\mathrm{div}_h J_h u_{\rm ini} = 0$

Then

- $\Pi_h u_h$ converges to $\overline{u}$ in $L^\infty(0, T; \mathbf{L}^2(\Omega))$
- $\nabla_h u_h$ converges to $\nabla\overline{u}$ in $\mathbf{L}^2(\Omega \times (0, T))^d$
- $\chi_h p_h$ weakly converges to $\overline{p}$ in $L^2(\Omega \times (0, T))$

Weak formulation :

$$
\begin{aligned}
&(\overline{u}, \overline{p}) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \\
&\eta \int_\Omega \overline{u} \cdot \overline{v} \,\mathrm{d}x + \int_\Omega \nabla \overline{u} : \nabla \overline{v} \,\mathrm{d}x + b(\overline{u}, \overline{u}, \overline{v}) - \int_\Omega \overline{p} \,\mathrm{div} \overline{v} \,\mathrm{d}x = \int_\Omega f \cdot \overline{v} \,\mathrm{d}x \\
&\forall \overline{v} \in \mathbf{H}^1_0(\Omega) \\
&\mathrm{div}(\overline{u}) = 0 \text{ a.e. in } \Omega.
\end{aligned}
$$

with $$b(\overline{u}, \overline{v}, \overline{w}) = \sum_{i=1}^{d} \sum_{j=1}^{d} \int_\Omega \overline{u}^{(i)} \partial_{(i)} \overline{v}^{(j)} \overline{w}^{(j)} \,\mathrm{d}x$$

Numerical scheme :

$$
\begin{aligned}
&(u, p) \in X_{h,0} \times Y_{h,0} \\
&\eta \int_\Omega \Pi_h u \cdot \Pi_h v \,\mathrm{d}x + \int_\Omega \nabla_h u : \nabla_h v \,\mathrm{d}x + b_h(u, u, v) \\
&\qquad - \int_\Omega \chi_h p \,\mathrm{div}_h v \,\mathrm{d}x = \int_\Omega f \cdot \Pi_h v \,\mathrm{d}x, \quad \forall v \in X_{h,0} \\
&\int_\Omega \mathrm{div}_h u \; \chi_h q \,\mathrm{d}x = 0, \quad \forall q \in Y_{h,0}
\end{aligned}
$$

Specifications on the numerical scheme for providing convergence

1. new coercivity condition $\|\Pi_h u\|_{\mathbf{L}^p} \leq C\|\nabla_h u\|_{\mathbf{L}^2}$, $p = 6$ β_h bounded by below
2. consistency : $\forall \varphi \in \mathbf{H}^1_0(\Omega)$, $S_h(\varphi) \to 0$, $\forall \psi \in L^2(\Omega)$, $\widetilde{S}_h(\psi) \to 0$
3. limit-conformity : $\forall (\varphi, \psi) \in Z(\Omega)$, $\overline{W}_h(\varphi, \psi) \to 0$
4. compactness : bounded sequence relatively compact in $\mathbf{L}^2$

Sufficient conditions on b_h for analysis

1. $b_h(u, u, u) \geq 0$
2. $b_h(u, v, w) \leq C\|\Pi_h u\|_{\mathbf{L}^4}\|\nabla_h v\|_{\mathbf{L}^2}\|\Pi_h w\|_{\mathbf{L}^4}$
3. if
   $\|\nabla_h u_h\|_{L^2}$ bounded and $\Pi_h u_h \to \overline{u} \in H^1_0(\Omega)$ in $L^2(\Omega)$
   $\|\nabla_h w_h\|_{L^2}$ bounded and $\Pi_h w_h \to \overline{w} \in H^1_0(\Omega)$ in $L^2(\Omega)$
   $\nabla_h v_h \rightharpoonup \nabla \overline{v}$ in $L^2(\Omega)$ with $\overline{v} \in H^1_0(\Omega)$
   then
   $b_h(u_h, v_h, w_h) \to b(\overline{u}, \overline{v}, \overline{w})$

case of centered $b_h(u, v, w)$ using natural definition and antisymmetry

$$\widehat{b}_h(u, v, w) = \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{\Omega} \Pi_h^{(i)} u \; \nabla_h^{(i,j)} v \; \Pi_h^{(j)} w \, \mathrm{d}x$$

$$b_h(u, v, w) = \frac{1}{2}(\widehat{b}_h(u, v, w) - \widehat{b}_h(u, w, v))$$

implies $b_h(u, v, w) = -b_h(u, w, v)$

case of centered $b_h(u, v, w)$ computed from face velocities and

$$\mathrm{div}_K u = \tfrac{1}{|K|} \textstyle\sum_{\sigma \in \mathcal{E}_K} |\sigma| u_\sigma n_\sigma \cdot n_{K,\sigma} = 0$$

$$b_h(u, v, w) = \textstyle\sum_{\substack{\sigma \in \mathcal{F} \\ \sigma = K|L}} |\sigma| u_\sigma \frac{\Pi_K v + \Pi_L v}{2} \cdot (\Pi_K w - \Pi_L w)$$

implies $b_h(u, v, w) = -b_h(u, w, v)$

case of upstream $b_h(u, v, w)$ computed from face velocities and

$$\mathrm{div}_K u = \tfrac{1}{|K|} \textstyle\sum_{\sigma \in \mathcal{E}_K} |\sigma| u_\sigma n_\sigma \cdot n_{K,\sigma} = 0$$

$$b_h(u, v, w) = \sum_{\substack{\sigma \in \mathcal{F} \\ \sigma = K|L}} |\sigma| (u_\sigma^+ \Pi_K v - u_\sigma^- \Pi_L v) \cdot (\Pi_K w - \Pi_L w)$$

implies $b_h(u, u, u) \geq 0$

estimate on the velocity

$v = u_h$ and $q = p_h$

$$\eta \|\Pi_h u_h\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla_h u_h\|^2_{\mathbf{L}^2} + \underbrace{b_h(u_h, u_h, u_h)}_{\geq 0} - \underbrace{\int_\Omega \chi_h p_h \, \mathrm{div}_h u_h \, dx}_{=0} = \int_\Omega f \cdot \Pi_h u_h \, dx.$$

$$\eta \|\Pi_h u_h\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla_h u_h\|^2_{\mathbf{L}^2} \leq \|f\|_{\mathbf{L}^2(\Omega)} \|\Pi_h u_h\|_{\mathbf{L}^2(\Omega)}$$

coercivity and Young inequalities lead to estimate on $\|\nabla_h u_h\|_{\mathbf{L}^2}$

estimate on the pressure

Let $v \in X_{h,0}$ such that $\|v\|_h = 1$ and $\beta_h \|\chi_h p\|_{L^2(\Omega)} \leq \int_\Omega \chi_h p \,\mathrm{div}_h v \; dx$

$$\beta_h \|\chi_h p\|_{L^2(\Omega)} \leq \eta \int_\Omega \Pi_h u_h \cdot \Pi_h v \; dx + \int_\Omega \nabla_h u_h : \nabla_h v \; dx + b_h(u_h, u_h, v) - \int_\Omega f \cdot \Pi_h v \; dx$$

Using Cauchy-Schwarz and hypotheses on b_h, we deduce :

$$\beta_h \|\chi_h p_h\|_{L^2(\Omega)} \leq \eta C_h \|\Pi_h u_h\|_{\mathbf{L}^2(\Omega)} + \|u_h\|_h + |b_h(u_h, u_h, v)| + C_h \|f\|_{\mathbf{L}^2(\Omega)}$$

$$|b_h(u_h, u_h, v)| \leq \|\Pi_h u\|_{\mathbf{L}^4(\Omega)} \|u_h\|_h \|\Pi_h v\|_{\mathbf{L}^4(\Omega)}$$

coercivity hypothesis : $\|\Pi_h v\|_{\mathbf{L}^4(\Omega)} \leq C_h \|v\|_h$ and $\|\Pi_h u_h\|_{\mathbf{L}^4(\Omega)} \leq C_h \|u_h\|_h$

$$|b_h(u_h, u_h, v)| \leq C_h^2 \|u_h\|_h^2$$

get $\|\chi_h p_h\|_{L^2(\Omega)} \leq C$

Existence of a discrete solution by topological degree argument
$F : X_{h,0} \times Y_{h,0} \times [0,1] \mapsto X_{h,0} \times Y_{h,0}$ such that,
for a given $(u_h, p_h) \in X_{h,0} \times Y_{h,0}$ and $\lambda \in [0,1]$, $(\tilde{u}, \tilde{p}) = F(u_h, p_h, \lambda)$ is defined by

$$\int_\Omega \Pi_h \tilde{u} \cdot \Pi_h v \; dx = \eta \int_\Omega \Pi_h u_h \cdot \Pi_h v \; dx + \int_\Omega \nabla_h u_h : \nabla_h v \; dx + \lambda b_h(u_h, u_h, v)$$
$$- \int_\Omega \chi_h p_h \mathrm{div}_h v \; dx - \int_\Omega \left(f \cdot \Pi_h v + G : \nabla_h v\right) \; dx \text{ for all } v \in X_{h,0},$$

and

$$\int_\Omega \chi_h \tilde{p} \, \chi_h q \; dx = \int_\Omega \chi_h p_h \mathrm{div}_h u_h \; dx \text{ for all } q \in Y_{h,0}.$$

1. F continuous
2. $\forall (u_h, p_h)$, such that $F(u_h, p_h, \lambda) = (0,0)$, estimates on u_h and $\overline{p}_h$ independent of λ
3. for $\lambda = 0$ linear pb
4. for $\lambda = 1$ scheme

thus exists at least one solution

existence of $\overline{u} \in \mathbf{L}^2(\Omega)$, $\zeta \in \mathbf{L}^2(\Omega)^d$ and $\gamma \in L^2(\Omega)$ such that $\Pi_h u_h \to \overline{u}$ in $\mathbf{L}^2(\Omega)$
$\nabla_h u_h \rightharpoonup \zeta$ in $\mathbf{L}^2(\Omega)^d$ and $\mathrm{div}_h u_h \rightharpoonup \gamma$ weakly in $L^2(\Omega)$
existence of $\overline{p} \in L_0^2(\Omega)$ such that $\chi_h p_h \to \overline{p}$ weakly in $L^2(\Omega)$

Regularity of the limit :

$$\int_\Omega (\nabla_h u_h : \varphi + \Pi_h u_h \cdot \mathrm{div}\varphi)\, \mathrm{d}\mathbf{x} \le \overline{W}_h(\varphi, 0) \|u_h\|_h$$

implies

$$\int_\Omega (\zeta : \varphi + \overline{u} \cdot \mathrm{div}\varphi)\, \mathrm{d}\mathbf{x} = 0 \text{ and } \zeta = \nabla \overline{u},\ \overline{u} \in \mathbf{H}_0^1(\Omega)$$

$$\int_\Omega (\Pi_h u_h \cdot \nabla\psi + \mathrm{div}_h u_h \psi)\, \mathrm{d}\mathbf{x} \le \overline{W}_h(0, \psi) \|u_h\|_h.$$

Let $\widetilde{I}_h : L_0^2(\Omega) \mapsto Y_{h,0}$ be defined by $\widetilde{I}_h \psi = \underset{z \in Y_{h,0}}{\mathrm{argmin}} \|\chi_h z - \psi\|_{L^2(\Omega)}$

Then

$$\int_\Omega \Pi_h u_h \cdot \nabla\psi\, \mathrm{d}\mathbf{x} \le \|\mathrm{div}_h u_h\|_{L^2(\Omega)} \|\chi_h \widetilde{I}_h \psi - \psi\|_{L^2(\Omega)} + \overline{W}_h(0, \psi) \|u_h\|_h.$$

implies $\int_\Omega \overline{u} \cdot \nabla\psi\, \mathrm{d}\mathbf{x} = 0$ for all $\psi \in H^1(\Omega)$ which implies that $\mathrm{div}\overline{u} = 0$

proof that $(\overline{u}, \overline{p})$ is weak solution to NS :

$w \in \mathbf{H}_0^1(\Omega)$, take $v = I_h w$ in scheme
$T_1 + T_2 + T_3 - T_4 = T_5$ passing to the limit as $h \to 0$ gives

$$T_1 = \eta \int_\Omega \Pi_h u_h \cdot \Pi_h I_h w \, dx \to \eta \int_\Omega \overline{u} \cdot w \, dx$$

$$T_2 = \int_\Omega \nabla_h u_h : \nabla_h I_h w \, dx \to \int_\Omega \nabla \overline{u} : \nabla w \, dx$$

$$T_3 = b_h(u_h, u_h, I_h w) \to b(\overline{u}, \overline{u}, w)$$

$$T_4 = \int_\Omega \chi_h p_h \mathrm{div}_h I_h w \, dx \to \int_\Omega \overline{p} \, \mathrm{div} w \, dx$$

$$T_5 = \int_\Omega f \cdot \Pi_h I_h w \, dx \to \int_\Omega f \cdot w \, dx$$

strong convergence of $\nabla_h u_h$

Taking as test function $v = u_h$ and passing to the supremum limit as $h \to 0$

$$\limsup_{h\to 0} \left(\eta \|\Pi_h u_h\|_{\mathbf{L}^2(\Omega)} + \|u_h\|_h\right) \leq \int_\Omega f \cdot \overline{u} \; dx$$

choosing $\overline{v} = \overline{u}$ as test function in continuous weak formulation

$$\eta \|\overline{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \overline{u}\|_{\mathbf{L}^2(\Omega)^d} = \int_\Omega f \cdot \overline{u} \; dx$$

we get

$$\limsup_{h\to 0} \left(\eta \|\Pi_h u_h\|_{\mathbf{L}^2(\Omega)} + \|u_h\|_h\right) \leq \eta \|\overline{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \overline{u}\|_{\mathbf{L}^2(\Omega)^d}$$

by weak limits

$$\liminf_{h\to 0} \left(\eta \|\Pi_h u_h\|_{\mathbf{L}^2(\Omega)} + \|u_h\|_h\right) \geq \eta \|\overline{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \overline{u}\|_{\mathbf{L}^2(\Omega)^d}$$

this implies $\|\nabla_h u_h\|_{\mathbf{L}^2(\Omega)^d} \to \|\nabla \overline{u}\|_{\mathbf{L}^2(\Omega)^d}$

Navier-Stokes, transient case

$$\begin{aligned} \partial_t \overline{u} - \Delta \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} &= f && \text{in } \Omega \times (0, T) \\ \mathrm{div}\overline{u} &= 0 && \text{in } \Omega \times (0, T) \\ \overline{u} &= 0 && \text{on } \partial\Omega \times (0, T) \\ \overline{u}(\cdot, 0) &= u_{\mathrm{ini}} && \text{in } \Omega \end{aligned}$$

$$\overline{u} \in L^2(0, T, E(\Omega)) \cap L^\infty(0, T, \mathbf{L}^2(\Omega))$$

$$\begin{aligned} &- \int_0^T \int_\Omega \overline{u}(x,t) \cdot \partial_t \bar{v}(x,t)\, dxdt - \int_0^T \int_\Omega \overline{u}_{\mathrm{ini}}(x) \cdot \bar{v}(x,0)\, dx \\ &\qquad + \int_0^T \int_\Omega \nabla \overline{u}(x,t) : \nabla \bar{v}(x,t)\, dxdt + \int_0^T b(\overline{u}(\cdot,t), \overline{u}(\cdot,t), \bar{v}(\cdot,t))\, dt \\ &\qquad\quad = \int_0^T \int_\Omega f(x,t) \cdot \bar{v}(x,t)\, dxdt \\ &\forall \bar{v} \in L^2(0, T, E(\Omega)) \cap \mathbf{C}_c^\infty(\Omega \times (-\infty, T)) \end{aligned}$$

$$u_h = (u_h^{(n)})_{n=0,\dots,N}\,,\ p_h = (p_h^{(n)})_{n=1,\dots,N} \text{ such that } u_h^{(0)} = J_h u_{\rm ini}$$
and, $\forall n = 0, \dots, N-1$:

$$u_h^{(n+1)} \in X_{h,0}\,,\ p_h^{(n+1)} \in Y_{h,0}\,,$$

$$\int_\Omega \Pi_h \delta_h^{n+\frac{1}{2}} u_h \cdot \Pi_h v \mathrm{d}x + \int_\Omega \nabla_h u_h^{(n+1)} : \nabla_h v \mathrm{d}x + b_h(u^{n+1}, u^{n+1}, v)$$

$$- \int_\Omega \chi_h p_h^{(n+1)} \mathrm{div}_h v \mathrm{d}x = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h v \mathrm{d}x\mathrm{d}t, \quad \forall v \in X_{h,0}$$

$$\int_\Omega \mathrm{div}_h u_h^{(n+1)} \chi_h q \mathrm{d}x = 0, \quad \forall q \in Y_{h,0}$$

Specifications on the numerical scheme for providing convergence

1. coercivity : $\|\Pi_h u\|_{\mathbf{L}^p} \le C \|\nabla_h u\|_{\mathbf{L}^2}$, $p = 6$ and β_h bounded by below
2. consistency : $\forall \varphi \in \mathbf{H}_0^1(\Omega)$, $S_h(\varphi) \to 0$, $\forall \psi \in L^2(\Omega)$, $\widetilde{S}_h(\psi) \to 0$
3. limit-conformity : $\forall (\varphi, \psi) \in Z(\Omega)$, $\overline{W}_h(\varphi, \psi) \to 0$
4. compactness : bounded sequence relatively compact in $\mathbf{L}^2$
5. for all $\varphi \in \mathbf{L}^2(\Omega)$, $\Pi_h J_h \varphi \to \varphi$ in $\mathbf{L}^2(\Omega)$
6. $\delta t_h \to 0$ as $h \to 0$

Sufficient conditions on b_h for analysis

1. $b_h(u,u,u) \geq 0$
2. $b_h(u,v,w) \leq C\|\Pi_h u\|_{L^4}\|\nabla_h v\|_{L^2}\|\Pi_h w\|_{L^4}$
3. if
   $\|\nabla_h u_h\|_{L^2(0,T;L^2)}$ bounded and $\Pi_h u_h \to \overline{u} \in L^2(0,T;H_0^1(\Omega))$ in $L^2(0,T;L^2)$
   $\|\nabla_h w_h\|_{L^\infty(0,T;L^2)}$ bounded and $\Pi_h w_h \to \overline{w} \in L^\infty(0,T;H_0^1(\Omega))$ in $L^\infty(0,T;L^2)$
   $\nabla_h v_h \rightharpoonup \nabla\overline{v}$ in $L^2(\Omega\times]0,T[)$ with $\overline{v} \in H_0^1(\Omega)$
   then
   $$\int_0^T b_h(u_h,v_h,w_h)\mathrm{d}t \to \int_0^T b(\overline{u},\overline{v},\overline{w})\mathrm{d}t$$

Estimates on the velocity and its discrete gradient

$v = \delta t^{n+\frac{1}{2}} u_h^{(n+1)}$ and $q = p_h^{(n+1)}$, since $b_h(u_h^{(n+1)}, u_h^{(n+1)}, u_h^{(n+1)}) \geq 0$, we get

$$\int_\Omega \left(\Pi_h u_h^{(n+1)} - \Pi_h u_h^{(n)}\right) \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_h u_h^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x\mathrm{d}t$$

$$\frac{1}{2} \int_\Omega \left[\left|\Pi_h u_h^{(n+1)}\right|^2 - \left|\Pi_h u_h^{(n)}\right|^2\right] \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_h u_h^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h u_h^{(n+1)} \mathrm{d}x\mathrm{d}t$$

$$\frac{1}{2} \|\Pi_h u_h^{(m)}\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2} \|\Pi_h u^{(0)}\|^2_{\mathbf{L}^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_h u_h\|^2_{\mathbf{L}^2(\Omega)^d} \mathrm{d}t \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_h u_h \mathrm{d}x\mathrm{d}t$$

proves

$$\int_0^{t^{(m)}} \int_\Omega |\nabla_h u_h^{(m)}|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_\Omega \left|\Pi_h u_h^{(m)}\right|^2 \mathrm{d}x \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_h u_h^{(m)} \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_\Omega |\Pi_h J_h u_{\mathrm{ini}}|^2 \mathrm{d}x$$

implies $\|\Pi_h u_h\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C$ and $\|\nabla_h u_h\|_{\mathbf{L}^2(\Omega\times(0,T))^d} \leq C$

leads to existence of discrete solution by topological degree method
and leads, similarly to steady Navier-Stokes problem, to weak compactness
$\Pi_h u_h \rightharpoonup \overline{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$
$\nabla_h u_h \rightharpoonup \zeta$ in $\mathbf{L}^2(\Omega \times (0, T))^d$
then $\overline{u} \in L^2(0, T; \mathbf{H}^1_0(\Omega))$ with free-divergence and $\zeta = \nabla u$
for strong convergence, compactness in space not sufficient

Estimates on the discrete time derivative of the velocity

$|w|_{\star,h} = \sup \left\{ \int_\Omega \Pi_h w \cdot \Pi_h v \mathrm{d}x \ : v \in E_h, \|v\|_h = 1 \right\}$ where
$E_h = \{v \in X_{h,0}, \mathrm{div}_h v = 0\}$
let $v \in E_h$ such that $\|v\|_h = 1$

$$\int_\Omega \Pi_h \delta_h^{n+\frac{1}{2}} u_h \cdot \Pi_h v \mathrm{d}x = -\int_\Omega \nabla_h u_h^{(n+1)} : \nabla_h v \mathrm{d}x - b_h(u^{n+1}, u^{n+1}, v)$$
$$+ \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h v \mathrm{d}x \mathrm{d}t$$

gives, using Young inequality and properties of b_h,

$$|\delta_h u_h|_{\star,h} \leq C(1 + \|u^{(n+1)}\|_h^2) + \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |f|^2 \mathrm{d}x \mathrm{d}t$$

multiplying by $\delta t^{n+\frac{1}{2}}$ and summing over n gives $\displaystyle\int_0^T |\delta_h u_h|_{\star,h} \mathrm{d}t \leq C$

discrete Aubin-Simon theorem

$T > 0$ and B Banach space $(B_m)_{m\in\mathbb{N}}$ finite dimensional subspaces of B
$t_m^{(0)} = 0 < t_m^{(1)} < \ldots < t_m^{(N_m)} = T$ and $\delta t_m^{(n)} = t_m^{(n)} - t_m^{(n-1)}$
$\{u_m^{(n)}, n = 0, \ldots, N_m\} \subset B_m$ and $u_m(\cdot, t) = (1 - \alpha_m^{(n)})u_m^{(n-1)} + \alpha_m^{(n)} u_m^{(n)} \in B_m$

$$\delta_m u_m(\cdot, t) = \delta_m^{(n)} u_m := \frac{1}{\delta t_m^{(n)}}(u_m^{(n)} - u_m^{(n-1)}) \text{ for a.e. } t \in (t_m^{(n-1)}, t_m^{(n)}), n \in \{1, \ldots, N_m\}.$$

$X_m = (B, \|\cdot\|_{X_m})$ and $Y_m = (B, \|\cdot\|_{Y_m})$

(h1) For any $(w_m)_{m\in\mathbb{N}}$ with $w_m \in B_m$ and $\|w_m\|_{X_m} \leq C$ then exists $w \in B$ and subsequence such that $w_m \to w$ in B as $m \to +\infty$

(h2) For any $(w_m)_{m\in\mathbb{N}}$ such that $w_m \in B_m$, $\|w_m\|_{X_m} \leq C$ and exists $w \in B$ such that $w_m \to w$ in B and $\|w_m\|_{Y_m} \to 0$ as $m \to +\infty$, then $w = 0$

(h3) $\alpha_m^{(n)} \leq C$ and $\|u_m\|_{L^1(0,T;X_m)} \leq C$

(h4) $\|\delta_m u_m\|_{L^1(0,T;Y_m)} \leq C$

Then exists $u \in L^1(0, T; B)$ and subsequence s.t. $u_m \to u$ in $L^1(0, T; B)$

application of discrete Aubin-Simon theorem for convergence in $L^1(0, T; L^2(\Omega))$

$B = \mathbf{L}^2(\Omega)$ and $B_h = \{\Pi_h v, v \in E_h\}$
$\| \cdot \|_{X_h} = \|\nabla_h \cdot \|_{L^2(\Omega)}$
$\| \cdot \|_{Y_h} = | \cdot |_{\star,h}$

(h_1) : compactness hypothesis

(h_2) : let $(v_h)_{h\in\mathcal{D}}$ such that $\nabla_h v_h$ bounded, $\Pi_h v_h \to v$ in $\mathbf{L}^2(\Omega)$ and $|v_h|_{\star,h} \to 0$ as $h \to 0$ $\int_\Omega \Pi_h v_h \cdot \Pi_h v_h \, dx \leq |v_h|_{\star,h} \|v_h\|_h \to 0$ as $h \to 0$ implies $v = 0$

(h_3) and (h_4) : estimates

implies that there exists $\overline{u} \in L^1(0, T, \mathbf{L}^2(\Omega))$ and subsequence such that $\Pi_h u_h \to \overline{u}$ in $L^1(0, T, \mathbf{L}^2(\Omega))$ as $h \to 0$

$\theta \in C_c^\infty([0,T))$ and $w \in E(\Omega)$ $w_h \in X_{h,0}$ with $\int_\Omega \chi_h q \mathrm{div}_h w_h = 0$ for all $q \in Y_{h,0}$,

$\Pi_h w_h \to w$ in $\mathbf{L}^2(\Omega)$ and $\nabla_h w_h \to \nabla w$ in $\mathbf{L}^2(\Omega)^d$ (implies $\Pi_h w_h \to w$ in $\mathbf{L}^4(\Omega)$)

$v = \delta t^{(n+\frac{1}{2})}\theta(t^{(n)}) w_h$ gives $T_1 + T_2 + T_3 + T_4 = T_5$ with

$$T_1 = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \Pi_h \delta^{(n+\frac{1}{2})} u_h \cdot \Pi_h w_h \mathrm{d}x$$

$$T_2 = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \nabla_h u_h^{(n+1)} : \nabla_h w_h \mathrm{d}x$$

$$T_3 = -\sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \chi_h p_h^{(n+1)} \mathrm{div}_h w_h \mathrm{d}x$$

$$T_4 = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} b_h(u^{(n+1)}, u^{(n+1)}, w_h) dt$$

$$T_5 = \sum_{n=0}^{N-1} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_h w_h \mathrm{d}x \mathrm{d}t$$

convergence of all terms except T_4 similar to transient Stokes problem

1. many numerical schemes available
2. good night, see you !