Relative entropy based error estimate for the finite volume approximation of strong solutions to systems of conservation laws

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Outline of the talk

1. Hyperbolic systems: strong and weak entropy solution
   - Definition of the problem
   - Relative entropy and weak-strong uniqueness principle
2. Finite Volume approximations and their stability
   - The Finite Volume scheme
   - Stability results
3. Error estimate for strong solutions
   - Continuous formulations for the discrete solution
   - Error estimate based on the relative entropy
4. Conclusion and prospects

Outline of the talk

Hyperbolic system of conservation laws

Consider the Cauchy problem

$$\begin{cases} \partial_t \mathbf{u} + \sum_{\alpha=1}^{d} \partial_\alpha \mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0} & \text{for } (\boldsymbol{x}, t) \in \mathbb{R}^d \times (0, \infty), \\ \mathbf{u}(\boldsymbol{x}, 0) = \mathbf{u}_0(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathbb{R}^d, \end{cases} \tag{1}$$

where

- $\mathbf{u} \in \Omega$ (set of the admissible states) convex bounded subset of $\mathbb{R}^m$;
- $\mathbf{f} = (\mathbf{f}_1, \dots \mathbf{f}_d) \in C^2(\overline{\Omega}; \mathbb{R}^{m \times d})$;
- $\mathbf{f}_\alpha'(\mathbf{u}) \in \mathcal{M}_m(\mathbb{R})$ is diagonalizable with real eigenvalues for all $\mathbf{u} \in \Omega$.

Strong solution to (1)

$\mathbf{u} \in C^1(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ is a strong solution if it satisfies (1).

Entropy/entropy flux pair

We assume that there exists an entropy $\eta \in C^2(\overline{\Omega}; \mathbb{R}_+)$ such that

- η is uniformly convex:

$$\mathrm{spec}\,(\eta''(\mathbf{u})) \subset [\beta, M], \qquad \forall \mathbf{u} \in \overline{\Omega};$$

- there exists an entropy flux $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) : \mathbb{R}^m \to \mathbb{R}^d$ such that

$$\xi_\alpha'(\mathbf{u}) = \eta'(\mathbf{u})\mathbf{f}_\alpha'(\mathbf{u}), \qquad \forall \alpha \in \{1, \dots, d\}.$$

Strong solutions are isentropic

Let $\mathbf{u}$ be a strong solution, then

$$\partial_t \eta(\mathbf{u}) + \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(\mathbf{u}) = 0 \quad \text{on } \mathbb{R}^d \times (0, \infty).$$

Entropy Weak solutions

Entropy weak solutions

$\mathbf{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+; \Omega)$ is said to be an entropy weak solution if

1. it is a weak solution, i.e.,

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m);$$

2. it dissipates entropy:

$$\partial_t \eta(\mathbf{u}) + \nabla \cdot \boldsymbol{\xi}(\mathbf{u}) \le 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+).$$

Strong solutions

- are entropy weak solutions
- may develop discontinuities after a finite time
- global existence with appropriate relaxation [Hanouzet-Natalini'03, Yong'04]

Entropy weak solutions exist and are unique if

- $m = 1$: scalar conservation laws [Vol'pert'67, Kružkov'70]
- $d = 1$ and small data [Bianchini-Bressan'05]
- $\mathbf{f}_\alpha(\mathbf{u}) = \mathbb{A}_\alpha \mathbf{u}$ with $\mathbb{A}_\alpha$ symetric [Friedrichs'54]
- if there exists a strong solution [DiPerna'79, Dafermos'79]

Well-posedness may fail otherwise [De Lellis-Székelyhidi'09]

Relative entropy/relative entropy flux

Relative entropy

The relative entropy of $\mathbf{v}$ w.r.t. $\mathbf{u}$ is given by

$$H(\mathbf{v},\mathbf{u}) = \eta(\mathbf{v}) - \eta(\mathbf{u}) - \eta'(\mathbf{u})(\mathbf{v}-\mathbf{u}), \qquad \forall \mathbf{u},\mathbf{v} \in \Omega.$$

The corresponding relative entropy flux $\mathbf{Q} = (Q_1,\dots,Q_d) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^d$ is given by

$$Q_\alpha(\mathbf{v},\mathbf{u}) = \xi_\alpha(\mathbf{v}) - \xi_\alpha(\mathbf{u}) - \eta'(\mathbf{u})(\mathbf{f}_\alpha(\mathbf{v}) - \mathbf{f}_\alpha(\mathbf{u})), \qquad \forall \mathbf{u},\mathbf{v} \in \Omega.$$

One has

$$\frac{\beta}{2}|\mathbf{v}-\mathbf{u}|^2 \le H(\mathbf{v},\mathbf{u}) \le \frac{M}{2}|\mathbf{v}-\mathbf{u}|^2, \qquad \forall \mathbf{u},\mathbf{v} \in \Omega.$$

Relative entropy/relative entropy flux

Let $\mathbf{v}$ be an entropy weak solution and $\mathbf{u}$ be a strong solution, then

$$\partial_t H(\mathbf{v},\mathbf{u}) + \boldsymbol{\nabla}\cdot\mathbf{Q}(\mathbf{v},\mathbf{u}) \leq -\sum_{\alpha=1}^{d} \partial_\alpha \mathbf{u}\cdot\mathbf{Z}_\alpha(\mathbf{v},\mathbf{u})$$

where $\mathbf{Z}_\alpha(\mathbf{v},\mathbf{u}) \in \mathbb{R}^m$ satisfies

$$|\mathbf{Z}_\alpha(\mathbf{v},\mathbf{u})| \leq C_{\mathbf{Z}}|\mathbf{u}-\mathbf{v}|^2, \qquad \forall \mathbf{u},\mathbf{v}\in\Omega.$$

$$\mathbf{Z}_\alpha(\mathbf{v},\mathbf{u}) = \eta''(\mathbf{u})\left(\mathbf{f}_\alpha(\mathbf{v}) - \mathbf{f}_\alpha(\mathbf{u}) - \mathbf{f}'_\alpha(\mathbf{u})(\mathbf{v}-\mathbf{u})\right), \qquad \forall \mathbf{u},\mathbf{v}\in\Omega$$

Weak-strong uniqueness principle

Theorem ([DiPerna'79, Dafermos'79])

Let $\mathbf{u}$ *be a strong solution and* $\mathbf{v}$ *be a weak entropy solution, then*

$$\int_{|x|<r} |\mathbf{v}(x,T) - \mathbf{u}(x,T)|^2 \mathrm{d}x \leq C(\|\nabla \mathbf{u}\|_\infty, T) \int_{|x|<r+L_f T} |\mathbf{v}_0(x) - \mathbf{u}_0(x)|^2 \mathrm{d}x$$

Sketch of the proof:

- Equivalence $H(\mathbf{v},\mathbf{u}) \sim |\mathbf{u}-\mathbf{v}|^2$
- Gronwall Lemma
- Finite speed of propagation

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Finite Volume scheme

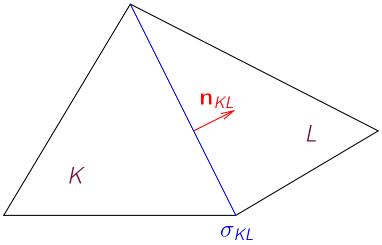
Time discretization: $t_n = n\Delta t$

Unstructured mesh: $\mathcal{T}$

- $h = \sup_{K\in\mathcal{T}} \text{diam}(K) < \infty$
- regular mesh

$$|K| \geq ah^d, \quad |\partial K| \leq h^{d-1}/a$$

- $\mathcal{N}_K$: neighboring cells of K
- $\mathcal{E}$: set of the edges



Finite Volume scheme

- Discrete initial data: $$\mathbf{u}_K^0 = \frac{1}{|K|}\int_K \mathbf{u}_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \in \mathbb{R}^m$$
- Discrete conservation laws: $$\mathbf{u}_K^{n+1} = \mathbf{u}_K^n - \frac{\Delta t}{|K|}\sum_{L\in\mathcal{N}_K} |\sigma_{KL}|\mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n)$$

Requirements on $\mathbf{F}_{KL}$ ([Bouchut'04])

(a) **Conservation:** $\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) + \mathbf{F}_{LK}(\mathbf{v},\mathbf{u}) = \mathbf{0}, \quad \forall \mathbf{u},\mathbf{v} \in \Omega$

(b) **Consistency:** $\mathbf{F}_{KL}(\mathbf{u},\mathbf{u}) = \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_{KL}, \quad \forall \mathbf{u} \in \Omega$

(c) **Stability**: for all $\lambda \geq \lambda^\star > 0$,

$$\mathbf{u} - \frac{1}{\lambda}\big(\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) - \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_{KL}\big) \in \Omega, \qquad \forall \mathbf{u},\mathbf{v} \in \Omega$$

(d) **Discrete entropy flux ξ_{KL}:** with

- $\xi_{KL}(\mathbf{u},\mathbf{v}) + \xi_{LK}(\mathbf{v},\mathbf{u}) = 0$,
- for all $\lambda \geq \lambda^\star > 0$,

$$\xi_{KL}(\mathbf{u},\mathbf{v}) - \boldsymbol{\xi}(\mathbf{u}) \cdot \mathbf{n}_{KL} \leq \lambda \left[\eta(\mathbf{u}) - \eta\left(\mathbf{u} - \frac{1}{\lambda}\left(\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) - \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_{KL}\right)\right)\right]$$

CFL condition ([Courant-Friedrichs-Lewy'28])

(e) **Time step restriction:** $\dfrac{\lambda^\star \Delta t}{a^2 h} \leq 1, \qquad \forall K \in \mathcal{T}.$

How to construct the numerical fluxes $\mathbf{F}_{KL}$

Riemann problem

The solution to the 1d Cauchy problem

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \left[\mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_{KL}\right] = 0 & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ \mathbf{u}(x,0) = \mathbf{u}_K \mathbf{1}_{x<0} + \mathbf{u}_L \mathbf{1}_{x>0} \end{cases}$$

is self-similar:

$$\mathbf{u}(x,t) = \mathcal{R}\left(\frac{x}{t}; \mathbf{u}_K, \mathbf{u}_L\right).$$

- Godunov scheme [Godunov'59]: exact resolution of the Riemann problem

$$\mathbf{F}_{KL}(\mathbf{u}_K, \mathbf{u}_L) = \mathbf{f}(\mathcal{R}(0; \mathbf{u}_K, \mathbf{u}_L)) \cdot \mathbf{n}_{KL}$$

- Approximate Riemann solvers [Harten-Lax-van Leer'83], [Toro'09], ...

Nonlinear stability results

Proposition (Nonlinear stability [Bouchut'04])

Under the assumptions on $\mathbf{F}_{KL}$ *and* Δt

- **Uniform stability**: $\mathbf{u}_K^n \in \Omega$ *for all* $K \in \mathcal{T}$ *and* $n \geq 0$
- **Consistency of the entropy fluxes:**
$$\xi_{KL}(\mathbf{u}, \mathbf{u}) = \boldsymbol{\xi}(\mathbf{u}) \cdot \mathbf{n}_{KL}, \qquad \forall \mathbf{u} \in \Omega.$$
- **Discrete entropy inequality**: $\forall K \in \mathcal{T}, \forall n \geq 0.$
$$\frac{\eta(\mathbf{u}_K^{n+1}) - \eta(\mathbf{u}_K^n)}{\Delta t}|K| + \sum_{L \in \mathcal{N}_K} |\sigma_{KL}|\, \xi_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) \leq 0.$$

A new stability estimate

Weak BV-estimate ([C.-Mathis-Seguin'15])

Under the strengthen CFL condition

$$\frac{M}{\beta}\frac{\lambda^\star \Delta t}{a^2 h} \le 1 - \zeta, \qquad \zeta \in (0,1),$$

there exists $C_{\mathrm{WBV}}(r, T, \zeta) > 0$ such that

$$\sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \left| \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) - \mathbf{f}(\mathbf{u}_K^n) \cdot \mathbf{n}_{KL} \right| \le \frac{C_{\mathrm{WBV}}}{\sqrt{h}}.$$

Consequences of the weak BV estimate

The weak BV estimate yields:

- the entropy fluxes based weak BV estimate:

$$\sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{T}_r} \sum_{L\in\mathcal{N}_K} |\sigma_{KL}| \left| \xi_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) - \boldsymbol{\xi}(\mathbf{u}_K^n) \cdot \mathbf{n}_{KL} \right| \leq \|\eta'\|_\infty \frac{C_{\mathrm{wBV}}}{\sqrt{h}},$$

- the time BV estimate on the solution...

$$\sum_{n\geq 0} \sum_{K\in\mathcal{T}_r} \left| \mathbf{u}_K^{n+1} - \mathbf{u}_K^n \right| |K| \leq \frac{C_{\mathrm{wBV}}}{\sqrt{h}},$$

- ...and on the entropy

$$\sum_{n\geq 0} \sum_{K\in\mathcal{T}_r} \left| \eta(\mathbf{u}_K^{n+1}) - \eta(\mathbf{u}_K^n) \right| |K| \leq \|\eta'\|_\infty \frac{C_{\mathrm{wBV}}}{\sqrt{h}}.$$

Related existing results

The scalar case $m = 1$:

- cartesian grids [Kuznetsov'76] (TVD schemes)
- unstructured meshes [Champier *et al.*'93], [Cockburn *et al.*'94], [Vila'94], [Eymard *et al.*'98], [Chainais-Hillairet'99], [Vovelle'02],...
- with stochastic forcing [Bauzet-Charrier-Gallouët'15]

Friedrichs systems $m > 1$, symmetric linear systems

- Godunov scheme [Rohde-Jovanovic'05]
- with constraints on the solution [Després-Lagoutière-Seguin'12]

Nonlinear systems $m > 1$

- $d \geq 1$ implicit FV scheme [Rohde-Jovanovic'06]
- $d \geq 1$ implicit DG scheme [Hiltebrand-Mishra'12]
- $d = 1$ ENO type scheme [Fjordholm-Käppeli-Mishra-Tadmor'14]

Link with numerical diffusion: heuristic

$\mathbf{u}_h \in C^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$ satisfies something like

$$\partial_t \mathbf{u}_h + \boldsymbol{\nabla} \cdot \mathbf{f}(\mathbf{u}_h) - h\zeta \Delta \mathbf{u}_h = \mathbf{0}.$$

Multiply by $\eta'(\mathbf{u}_h)$ and integrate on $B(0,r) \times (0,T)$, then

$$\int_{|\boldsymbol{x}|<r} \eta(\mathbf{u}_h(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x} + \int_0^T \int_{|\boldsymbol{x}|=r} \boldsymbol{\xi}(\mathbf{u}_h) \cdot \mathbf{n} \mathrm{d}\boldsymbol{x}\mathrm{d}t + h\zeta \int_0^T \int_{|\boldsymbol{x}|<r} \eta''(\mathbf{u}_h)\boldsymbol{\nabla}\mathbf{u}_h \cdot \boldsymbol{\nabla}\mathbf{u}_h \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq \int_{|\boldsymbol{x}|<r} \eta(\mathbf{u}_0)\mathrm{d}\boldsymbol{x}.$$

Therefore

$$\int_0^T \int_{|\boldsymbol{x}|<r} |\boldsymbol{\nabla}\mathbf{u}_h| \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq C(r,T) \left(\int_0^T \int_{|\boldsymbol{x}|<r} |\boldsymbol{\nabla}\mathbf{u}_h|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \right)^{1/2} \leq \frac{C(r,T,\zeta)}{\sqrt{h}}.$$

Why a restricted CFL: very simplified case

Linear transport equation $\partial_t u + a\partial_x u = 0$ with $a > 0$, **Upwind scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} h + a\left(u_i^n - u_{i-1}^n\right) = 0, \qquad \frac{a\Delta t}{h} = 1 - \zeta.$$

Multiply by $\Delta t u_i^n$ and sum over i and n:

$$T_1 = \frac{1}{2}\sum_{i=-L}^{L} h\left(\left(u_i^{N+1}\right)^2 - \left(u_i^0\right)^2 - \sum_{n=0}^{N}\left(u_i^{n+1} - u_i^n\right)^2\right),$$

$$T_2 = \frac{a}{2}\sum_{n=0}^{N}\Delta t\left(\left(u_L^n\right)^2 - \left(u_{-L-1}^n\right)^2 + \sum_{i=-L}^{L}\left(u_i^n - u_{i-1}^n\right)^2\right).$$

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Multiply by $\Delta t u_i^n$ and sum over i and n:

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$$T_2 = \frac{a}{2}\sum_{n=0}^{N}\Delta t\left(\left(u_L^n\right)^2 - \left(u_{-L-1}^n\right)^2 + \sum_{i=-L}^{L}\left(u_i^n - u_{i-1}^n\right)^2\right).$$

Numerical diffusion:

$$T_1 + T_2 = \mathcal{O}(1) + (a\zeta h)\sum_{n=0}^{N}\Delta t\sum_{i=-L}^{L}\left(\frac{u_i^n - u_{i-1}^n}{h}\right)^2 h.$$

Define the modified entropy flux X_{KL} by

$$X_{KL}(\mathbf{u},\mathbf{v}) := \boldsymbol{\xi}(\mathbf{u})\cdot\mathbf{n}_{KL} + \eta'(\mathbf{u})\left(\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) - \mathbf{f}(\mathbf{u})\cdot\mathbf{n}_{KL}\right)$$

Entropy dissipation through the edges

$$X_{KL}(\mathbf{u},\mathbf{v}) + X_{LK}(\mathbf{v},\mathbf{u}) \geq \frac{\beta}{2\lambda^\star}\left|\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) - \mathbf{f}(\mathbf{u})\cdot\mathbf{n}_{KL}\right|^2 + \frac{\beta}{2\lambda^\star}\left|\mathbf{F}_{LK}(\mathbf{v},\mathbf{u}) - \mathbf{f}(\mathbf{v})\cdot\mathbf{n}_{LK}\right|^2, \qquad \forall(\mathbf{u},\mathbf{v})\in\Omega^2.$$

The proof relies on

- the conservatitivity of the entropy flux $\xi_{KL}(\mathbf{u},\mathbf{v}) + \xi_{LK}(\mathbf{v},\mathbf{u}) = 0$
- Bouchut's condition

$$\xi_{KL}(\mathbf{u},\mathbf{v}) - \boldsymbol{\xi}(\mathbf{u})\cdot\mathbf{n}_{KL} \leq \lambda^\star\left[\eta(\mathbf{u}) - \eta\left(\mathbf{u} - \frac{1}{\lambda^\star}\left(\mathbf{F}_{KL}(\mathbf{u},\mathbf{v}) - \mathbf{f}(\mathbf{u})\cdot\mathbf{n}_{KL}\right)\right)\right]$$

- the uniform β-convexity of η.

Multiply the scheme

$$\frac{\mathbf{u}_K^{n+1} - \mathbf{u}_K^n}{\Delta t}|K| + \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n)$$

by $\eta'(\mathbf{u}_K)\Delta t$ and sum over $K \in \mathcal{T}_r$ and $n \in \{0, \dots, N_T\}$:

$$T_1 + T_2 = 0,$$

where

$$T_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \eta'(\mathbf{u}_K) \left(\mathbf{u}_K^{n+1} - \mathbf{u}_K^n\right) |K|$$

and

$$T_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \eta'(\mathbf{u}_K) \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n).$$

Since $\mathbf{u} \mapsto \eta(\mathbf{u}) - \frac{M}{2}|\mathbf{u} - \mathbf{u}_K^n|^2$ is concave, one has

$$T_1 \geq \underbrace{\sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} \left(\eta(\mathbf{u}_K^{n+1}) - \eta(\mathbf{u}_K^n)\right) |K|}_{T_{11}} \underbrace{- \frac{M}{2} \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}_r} |\mathbf{u}_K^{n+1} - \mathbf{u}_K^n|^2 |K|}_{T_{12}}.$$

- Positivity of η + Jensen inequality yield

$$T_{11} \geq - \int_{B(0,r)} \eta(\mathbf{u}_0) \mathrm{d}\boldsymbol{x} = \mathcal{O}(1).$$

- Using the scheme and

$$\sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \mathbf{f}(\mathbf{u}_K^n) \cdot \mathbf{n}_{KL} = \mathbf{f}(\mathbf{u}_K^n) \cdot \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \mathbf{n}_{KL} = \mathbf{0}, \qquad \forall K \in \mathcal{T}, \forall n \geq 0$$

together with Cauchy-Schwarz inequality and the regularity of the mesh:

$$T_{12} \geq - \frac{M \Delta t}{2a^2 h} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \left| \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) - \mathbf{f}(\mathbf{u}_K^n) \cdot \mathbf{n}_{KL} \right|^2$$

$$T_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \eta'(\mathbf{u}_K) \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n).$$

Using again that $\sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \mathbf{n}_{KL} = \mathbf{0}$, one gets

$$\begin{aligned} T_2 &= \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| X_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) \\ &= \sum_{n=0}^{N_T} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}_r} |\sigma_{KL}| \left(X_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) + X_{LK}(\mathbf{u}_L^n, \mathbf{u}_K^n) \right) + \mathcal{O}(1) \\ &\geq \frac{\beta}{2\lambda^\star} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}_r} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \left| \mathbf{F}_{KL}(\mathbf{u}_K^n, \mathbf{u}_L^n) - \mathbf{f}(\mathbf{u}_K^n) \cdot \mathbf{n}_{KL} \right|^2 + \mathcal{O}(1) \end{aligned}$$

Outline of the talk

Discrete solution

Definition (Discrete solution)

The discrete solution $\mathbf{u}_h \in BV_{loc}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$ *is defined by*

$$\mathbf{u}_h(\boldsymbol{x}, t) = \mathbf{u}_K^n \quad \text{if } (\boldsymbol{x}, t) \in K \times [t_n, t_{n+1}).$$

How far is $\mathbf{u}_h$ from being an entropy weak solution ?

$$\partial_t \mathbf{u}_h + \boldsymbol{\nabla} \cdot \mathbf{f}(\mathbf{u}_h) \overset{?}{=} \boldsymbol{\epsilon}(h) \qquad \partial_t \eta(\mathbf{u}_h) + \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(\mathbf{u}_h) \overset{?}{\leq} \epsilon'(h)$$

with $\boldsymbol{\epsilon}(h) \to \mathbf{0}$ and $\epsilon'(h) \to 0$ in some appropriate sense ? If so,

$$\partial_t H(\mathbf{u}_h, \mathbf{u}) + \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(\mathbf{u}_h, \mathbf{u}) \leq -\sum_{\alpha=1}^{d} \partial_\alpha \mathbf{u} \cdot \mathbf{Z}_\alpha(\mathbf{u}_h, \mathbf{u}) + \epsilon''(\mathbf{u}, h).$$

Discrete solution

Definition (Discrete solution)

The *discrete solution* $\mathbf{u}_h \in BV_{loc}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$ *is defined by*

$$\mathbf{u}_h(\boldsymbol{x}, t) = \mathbf{u}_K^n \quad \text{if } (\boldsymbol{x}, t) \in K \times [t_n, t_{n+1}).$$

How far is $\mathbf{u}_h$ from being an entropy weak solution ?

$$\partial_t \mathbf{u}_h + \boldsymbol{\nabla} \cdot \mathbf{f}(\mathbf{u}_h) \overset{?}{=} \boldsymbol{\epsilon}(h) \qquad \partial_t \eta(\mathbf{u}_h) + \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(\mathbf{u}_h) \overset{?}{\leq} \epsilon'(h)$$

with $\boldsymbol{\epsilon}(h) \to \mathbf{0}$ and $\epsilon'(h) \to 0$ in some appropriate sense ? If so,

$$\partial_t H(\mathbf{u}_h, \mathbf{u}) + \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(\mathbf{u}_h, \mathbf{u}) \leq -\sum_{\alpha=1}^{d} \partial_\alpha \mathbf{u} \cdot \mathbf{Z}_\alpha(\mathbf{u}_h, \mathbf{u}) + \epsilon''(\mathbf{u}, h).$$

Radon measures for errors terms

I. Weak formulation

Proposition (Weak formulation for the discrete solution)

Assume $\mathbf{u}_0 \in BV_{loc}(\mathbb{R}^d)$. *There exist positive Radon measures* $\mu_0 \in \left(C_c(\mathbb{R}^d)\right)'$ *and* $\mu \in \left(C_c(\mathbb{R}^d \times \mathbb{R}_+)\right)'$ *with*

$$\mu_0(B_d(\mathbf{0}, r)) \leq C(r)h, \qquad \mu(B_d(\mathbf{0}, r) \times [0, T]) \leq C(r, T)\sqrt{h} \qquad (*)$$

such that: $\forall \boldsymbol{\varphi} \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$,

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}_+} \mathbf{u}_h \partial_t \boldsymbol{\varphi} \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_{\mathbb{R}^d} \mathbf{u}_0 \boldsymbol{\varphi}(\cdot, 0) \mathrm{d}\boldsymbol{x} \right.$$
$$\left. + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \mathbf{f}(\mathbf{u}_h) \cdot \boldsymbol{\nabla} \boldsymbol{\varphi} \mathrm{d}\boldsymbol{x} \mathrm{d}t \right| \leq \langle \mu_0 \, ; |\boldsymbol{\varphi}(\cdot, 0)| \rangle + \langle \mu \, ; |\boldsymbol{\nabla} \boldsymbol{\varphi}| + |\partial_t \boldsymbol{\varphi}| \rangle \quad (**)$$

$(**)$ Calculations as in [Eymard *et al.*'98],...

$(*)$ $\mathbf{u}_0 \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ and weak BV estimate.

Radon measures for errors terms

II. Entropy inequality

Proposition (Entropy inequality for the discrete solution)

Assume $\mathbf{u}_0 \in BV_{loc}(\mathbb{R}^d)$*. There exist positive Radon measures* $\nu_0 \in \big(C_c(\mathbb{R}^d)\big)'$ *and* $\nu \in \big(C_c(\mathbb{R}^d \times \mathbb{R}_+)\big)'$ *with*

$$\nu_0(B_d(\mathbf{0}, r)) \leq C(r)h, \qquad \nu(B_d(\mathbf{0}, r) \times [0, T]) \leq C(r, T)\sqrt{h} \qquad (*)$$

such that: $\forall \varphi \in C^1_c(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$,

$$\iint_{\mathbb{R}^d \times \mathbb{R}_+} \eta(\mathbf{u}_h)\partial_t\varphi \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{\mathbb{R}^d} \eta(\mathbf{u}_0)\varphi(\cdot, 0)\mathrm{d}\boldsymbol{x} + \iint_{\mathbb{R}^d \times \mathbb{R}_+} \boldsymbol{\xi}(\mathbf{u}_h) \cdot \boldsymbol{\nabla}\varphi \mathrm{d}\boldsymbol{x}\mathrm{d}t \geq -\langle \nu_0 \,; |\varphi(\cdot, 0)|\rangle + \langle \nu \,; |\boldsymbol{\nabla}\varphi| + |\partial_t\varphi|\rangle \qquad (**)$$

$(**)$ Calculations as in [Eymard *et al.*'98],...

$(*)$ $\eta(\mathbf{u}_0) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and weak BV estimate.

Relative entropy estimate

Let $\mathbf{u}$ be a strong solution (isentropic) and $\mathbf{u}_h$ be the discrete solution, then, for all $\varphi \in C_c^1\left(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+\right)$,

$$\iint H(\mathbf{u}_h, \mathbf{u})\partial_t \varphi \mathrm{d}\boldsymbol{x}\mathrm{d}t + \iint \mathbf{Q}(\mathbf{u}_h, \mathbf{u}) \cdot \boldsymbol{\nabla}\varphi \mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$\geq - \iint \left(\sum_{\alpha=1}^{d} \partial_\alpha \mathbf{u} \cdot \mathbf{Z}_\alpha(\mathbf{u}_h, \mathbf{u}) \right) \varphi \mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$- \langle \nu_0; \varphi(0) \rangle - \langle \nu; |\boldsymbol{\nabla}\varphi| + |\partial_t \varphi| \rangle$$
$$- \langle \mu_0; |\boldsymbol{\psi}(0)| \rangle - \langle \mu; |\boldsymbol{\nabla}\boldsymbol{\psi}| + |\partial_t \boldsymbol{\psi}| \rangle$$

where $\boldsymbol{\psi} = \eta'(\mathbf{u})\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^m)$.

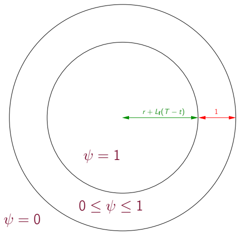
Choosing the test functions φ

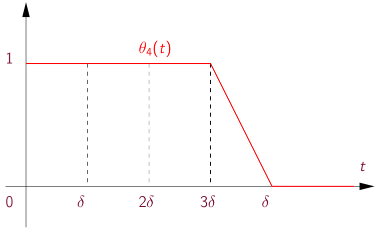
Finite speed of propagation

There exists $L_{\mathbf{f}}(\mathbf{f}, \eta, \Omega) \geq 0$ such that

$$L_{\mathbf{f}} H(\mathbf{u}_h, \mathbf{u}) + \sum_{\alpha=1}^{d} \frac{x_\alpha}{|\boldsymbol{x}|} Q_\alpha(\mathbf{u}_h, \mathbf{u}) \geq 0.$$

test functions: $\varphi_k(\boldsymbol{x}, t) = \theta_k(t)\psi(\boldsymbol{x}, t)$





Main result

Theorem ([C.-Mathis-Seguin'15])

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d \times \mathbb{R}_+$, then there exists C depending on $\mathcal{K}, \|\nabla u\|_\infty, \zeta, \dots$ such that

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\mathcal{K})} \leq Ch^{1/4}.$$

Main result

Theorem ([C.-Mathis-Seguin'15])

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d \times \mathbb{R}_+$, then there exists C depending on $\mathcal{K}, \|\nabla u\|_\infty, \zeta, \dots$ such that

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\mathcal{K})} \leq C h^{1/4}.$$

Under-optimal result. In practice,

- for strong solutions :

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\mathcal{K})} \leq C h$$

- for genuinely nonlinear problems

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\mathcal{K})} \leq C h$$

- in presence of contact discontinuities

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\mathcal{K})} \leq C h^{1/2}$$

Outline of the talk

Conclusions and Prospects

Conclusions:

- Weak BV estimate for nonlinear systems
- Error estimate (provided the L^∞ stability assumption) for FV

Prospects:

- Modeling error estimates [Mathis *et al.*'15], [Cancès *et al.*'15],
- Balancing numerical and modeling errors

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Prospects:

- Modeling error estimates [Mathis *et al.*'15], [Cancès *et al.*'15],
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Thank you for your attention !

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