Local/global well-posedness for incompressible Euler equations

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Plan

1. Structure
2. Tool
3. Local\global well-po
   - Local existence with Sub-critical regularitie $H^s, s > \frac{d}{2} + 1$
   - Global existence: 2d and $3d$ with axisymmetric initial data .
4. Critical regularitie $B_{2,1}^{\frac{d}{2}+1}$, Dini space

Euler Model 1755

$$(\mathcal{E})\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p = 0, & x\in\mathbb{R}^d,\ t\geq 0\\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{cases}$$

- Velocity field: $v = (v^1, .., v^d) \in \mathbb{R}^d$
- The o $v\cdot\nabla$ is defined by

$$v\cdot\nabla = \sum_{j=1}^d v^j\partial_j.$$

- The pre p is a scalar satisfying the elliptic equation

$$\Delta p = -\operatorname{div}(v\cdot\nabla v).$$

Leray's projector

- We define Leray's pro $\mathbb{P}$ by

$$\mathbb{P}v = v - \nabla\Delta^{-1}\mathrm{div}\, v$$

- If $\mathrm{div}\, v = 0$, then $\mathbb{P}v = v$.
- $\mathbb{P}(\nabla p) = 0$
- Euler equations can be written in the form

$$\partial_t v + \mathbb{P}\big(v \cdot \nabla v\big) = 0$$

Lagrangian formulation

♦ Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field in $L^1_T Lip(\mathbb{R}^d)$. We define the flow map $\psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\partial_t \psi(t,x) = v(t, \psi(t,x)), \quad \psi(0,x) = x$$

♦ For each $x \in \mathbb{R}^d$, the curve $t \mapsto \psi(t,x)$ is the trajectory of the particle located initially at x.

Some pro

1. For each $t \in [0, T]$, $\Psi(t) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism.
2. If $\operatorname{div} v = 0$ then $\psi(t)$ pre
3. Euler sy

$$\partial_{tt}\psi(t,x) = -(\nabla p)(t,\psi(t,x)), \qquad \det(\partial_x \psi(t,x)) = 1$$

Vorticity structure

- We recall that the vorticity ω is defined by

$$\omega = \mathrm{curl}\, v := \begin{cases} \partial_1 v^2 - \partial_2 v^1, & \text{if } d = 2, \\ \nabla \wedge v, & \text{if } d = 3. \end{cases}$$

- Biot-Savart law:

$$\Delta v = \begin{cases} \nabla^\perp \omega, & \text{if } d = 2, \\ \nabla \wedge \omega, & \text{if } d = 3. \end{cases}$$

Therefore in the plane

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy$$

Consequences

- Let $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ then for any $p \in]2, \infty]$

$$||v||_{L^p} \leq C||\omega||_{L^1 \cap L^\infty}$$

- If $\omega \in C_c(\mathbb{R}^2)$ and

$$v \in L^2(\mathbb{R}^2) \Longleftrightarrow \int_{\mathbb{R}^2} \omega dx = 0$$

Hint: we use that for $|y| \leq M, |x| \geq 2M$

$$\frac{1}{|x-y|^2} = \frac{1}{|x|^2} + O(|x|^{-3})$$

- There exist $C > 0$ such that for any $p \in]1, \infty[$

$$||\nabla v||_{L^p} \leq C \frac{p^2}{p-1} ||\omega||_{L^p}$$

Vorticity equation

- Helmholtz equation 1858 :

$$(d=2) \qquad \partial_t\omega + v\cdot\nabla\omega = 0$$

$$(d=3) \qquad \partial_t\omega + v\cdot\nabla\omega = \omega\cdot\nabla v$$

Conservation laws

- Energy

$$\int_{\mathbb{R}^d} \partial_t v \cdot v\, dx = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} (v \cdot \nabla v^j) v^j\, dx - \int_{\mathbb{R}^d} \nabla p \cdot v\, dx = \int_{\mathbb{R}^d} (\frac{1}{2}|v|^2 + p) \mathrm{div}\, v\, dx = 0$$

Thus

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}$$

L^p norms of ω

- In the plane we have

$$\omega(t,x) = \omega_0(\psi^{-1}(t,x)$$

Thus for any $p \in [1,\infty]$

$$\|\omega(t)\|_{L^p} = \omega_0\|_{L^p}$$

- A classical solution for Euler is a function $v \in C^1([0,T] \times \mathbb{R}^d)$ satisfying the equation in the usual sense.
- Given a Banach space $X \subset C_b^1$. We say that Euler is well-po X if for any $v_0 \in X$ with zero divergence, there exist $v \in \mathscr{C}([0,T]; X)$, for $T > 0$ and $v_0 \mapsto v$ is continuous.
- The life $T^\star$.

Hölder and Sobolev space

- Hölder space $n \in \mathbb{N}, s \in]0,1[$ then

$$u \in C^s(\mathbb{R}^d) \Longleftrightarrow \|u\|_{C^s} = \|u\|_{L^\infty} + \sup_{0<|x-y|\leq 1} \frac{|u(x)-u(y)|}{|x-y|^s} < \infty$$

$$u \in C^{n+s}(\mathbb{R}^d) \Longleftrightarrow \forall |\alpha| \leq n,\ \partial^\alpha u \in C^s$$

- Sobolev space $s \in \mathbb{R}$

$$u \in H^s(\mathbb{R}^d) \Longleftrightarrow \|u\|_s^2 := \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < \infty$$

- For $s > \frac{d}{2}$, H^s is an algebra:

$$\|uv\|_s \leq C\|u\|_s\|v\|_s$$

Be

- Dyadic partition of the unity:
  - There exist two radial functions $\chi \in \mathcal{D}(B)$ and $\varphi \in \mathcal{D}(C)$, with
$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d.$$
  - Littlewood-Paley o $\quad u \in \mathscr{S}'(\mathbb{R}^d)$
$$\Delta_{-1} u = \chi(\mathfrak{D})u, \quad \Delta_q u = \varphi(2^{-q}\mathfrak{D})u, \quad \forall q \in \mathbb{N}.$$
- For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$
$$u \in B^s_{p,r} \Longleftrightarrow 2^{js} \|\Delta_j u\|_{L^p} \in \ell^r(\{-1\} \cup \mathbb{N}).$$
- $H^s = B^s_{2,2}$; $\quad C^s = B^s_{\infty,\infty}$, $\quad$ if $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

Critical and sub-critical space

1. Sub-critical space X is said *sub – critical* if

$$\exists \varepsilon > 0 \text{ s. t. } X \subset C^{1+\varepsilon}$$

Examples: $H^s, s > \frac{d}{2} + 1$; $B^s_{p,r}, s > \frac{d}{p} + 1$; $C^s, s > 1$

2. Critical space $X \subset C^1_b$ is said *critical* if

$$\forall \varepsilon > 0, \quad X \not\subset C^{1+\varepsilon}$$

Examples: $B^{\frac{d}{2}+1}_{2,1}, B^1_{\infty,1}, B^{\frac{d}{p}+1}_{p,1}, ..$

Some well-posedness results

- Lichtenstein 1930:
  Local well-po **LWP**) in $C^\alpha, \alpha > 1, \quad \alpha \notin \mathbb{N}$
- Wölibber 1933: global existence in $2d$.
- Ebin-Marsden 1970: **LWP** in $H^s(M)$ with, $s > \frac{d}{2} + 1$ and M compact manifold.
- Bourguignon-Brezis1974: $W^{s,p}(\Omega)$, $s > \frac{d}{p} + 1$, Ω bounded domain.
- Kato-Ponce 1988: $H^s(\mathbb{R}^d), s > \frac{d}{2} + 1$.
- Beirao da Veiga 1984: global existence in critical space of Dini ty
- Vishik 1998: global existence in critical Be
- Chae 2004: **LWP** in critical Be $B^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$.

- Che

  when $v_0 \in C^\alpha, \alpha > 1$
- Misiolek-Yoneda 2014: we loo

  $v_0 \in C^1$.
- Bourgain-Li 2014: strong ill-po $H^{\frac{d}{2}+1}$.
- Elgindy-Masmoudi 2014: strong ill-po

  $C^m, m \in \mathbb{N}^\star$

Let $v_0 \in H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} + 1$. Then the sy
has a unique local solution $\mathscr{C}([0,T]; H^s)$, with

$$T \geq \frac{C}{\|v_0\|_s}.$$

Be $T^\star$ satisfie

$$T^* < +\infty \Longrightarrow \int_0^{T^*} \|\nabla v(t)\|_{L^\infty} dt = +\infty.$$

Blowup rate

Let $v_0 \in H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} + 1$ and assume that $T^\star < \infty$ then

1. There exist $C(s,d)$ such that

$$\|v(t)\|_s \geq \frac{C}{T^\star - t}$$

2. There exist $C_0 := C(\|v_0\|_{L^2})$ such that

$$\|v(t)\|_s \geq \frac{C_0}{(T^\star - t)^{\frac{s}{1+d/2}}}$$

Strategy of the proof:

1. A priori e
2. A (E_n): work with ODE.
3. Uniform e n in $\mathscr{C}([0,T]; H^s)$ with $T > 0$ inde n
4. Convergence to the limit sy
5. Uniquene

A priori e

- Recall the dyadic characterization of H^s

$$\|v\|_s^2 = \sum_{q\geq -1} 2^{2qs} \|\Delta_q v\|_{L^2}^2$$

- Denote by $v_q := \Delta_q$ then we localize the velocity equation

$$\partial_t v_q + v \cdot \nabla v_q + \nabla P_q = -[\Delta_q, v] \cdot \nabla v := C_q$$

with the notation

$$[\Delta_q, f]g = \Delta_q(fg) - f\Delta_q g$$

- Taking the L^2 inner product with v_q

$$\frac{1}{2}\frac{d}{dt}\|v_q(t)\|_{L^2}^2 + \int_{\mathbb{R}^d}(v\cdot\nabla v_q)\cdot v_q dx + \int_{\mathbb{R}^d}\nabla P_q\cdot v_q dx = \int_{\mathbb{R}^d} C_q\cdot v_q dx$$

Since div $v_q = 0$ then

$$\frac{1}{2}\frac{d}{dt}\|v_q(t)\|_{L^2}^2 = \int_{\mathbb{R}^d} C_q\cdot v_q dx$$

From Cauchy-Scwharz inequality

$$\frac{1}{2}\frac{d}{dt}\|v_q(t)\|_{L^2}^2 \leq \|v_q(t)\|_{L^2}\|C_q(t)\|_{L^2}$$

- Multiplying by 2^{2qs} ad summing over q we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 &\leq \sum_{q\geq -1} 2^{2qs}\|v_q(t)\|_{L^2}\|C_q(t)\|_{L^2}\\ &\leq \|v(t)\|_{H^s}\Big(\sum_{q\geq -1} 2^{2qs}\|C_q(t)\|_{L^2}^2\Big)^{\frac{1}{2}}\end{aligned}$$

- Commutator e

  $s > -1$

$$\Big(\sum_{q\geq -1} 2^{2qs}\|C_q(t)\|_{L^2}^2\Big)^{\frac{1}{2}} \leq C\|\nabla v(t)_{L^\infty}\|v(t)\|_{H^s}$$

- Therefore

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 \leq C\|\nabla v(t)\|_{L^\infty}\|v(t)\|_{H^s}^2$$

- Simplifying and integrating in time we get

$$\|v(t)\|_{H^s} \leq \|v_0\|_{H^s} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{H^s} d\tau.$$

- From Sobolev embedding we get for $s > \frac{d}{2} + 1$

$$f(T) := \sup_{0\leq t\leq T} \|v\|_{H^s} \leq \|v_0\|_{H^s} + C\,T\,f(T)^2$$

- Therefore, if $4CT\|v_0\|_{H^s} \leq 1$ then

$$f(T) \leq 2\|v_0\|_{H^s}$$

Improvement of the time existence

- We use the interpolation inequality (with $\theta = \frac{1+d/2}{s}$)

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq C\|v(t)\|_{L^2}^{1-\theta}\|v(t)\|_{H^s}^{\theta}\\ &\leq C\|v_0\|_{L^2}^{1-\theta}\|v(t)\|_{H^s}^{\theta}\end{aligned}$$

- Therefore we obtain

$$f(T) \leq \|v_0\|_{H^s} + C\|v_0\|_{L^2}^{1-\theta}\, T\, [f(T)]^{1+\theta}$$

- Therefore, if $C2^{1+\theta}\|v_0\|_{L^2}^{1-\theta}\|v_0\|_{H^s}^{\theta} T \leq 1$, then

$$f(T) \leq 2\|v_0\|_{H^s}$$

Blowup criterion

- If $T^\star < \infty$ then

$$\lim_{t \to T^\star} \|v(t)\|_{H^s} = +\infty$$

- From the energy e

$$\|v(t)\|_{H^s} \leq \|v_0\|_{H^s} e^{C \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}$$

- This implie

$$\int_0^{T^\star} \|\nabla v(\tau)\|_{L^\infty} d\tau = +\infty$$

Ao

- For $n \in \mathbb{N}$ we introduce the pro

$$\widehat{J_n v}(\xi) = \mathbf{1}_{B(0,2^n)} \widehat{v}(\xi)$$

- $J_n : H^s \to H^s$ is continuous uniformly on n
- Let v_n be the solution of

$$(\mathcal{E})_n \begin{cases} \frac{d}{dt} v_n = -J_n \mathbb{P}\big(J_n v_n \cdot \nabla J_n v_n\big) \\ v_{n|t=0} = J_n v_0. \end{cases}$$

- This is equivalent to the fixed point problem

$$v_n(t) = J_n v_0 + \int_0^t J_n \mathbb{P}\big(J_n v_n \cdot \nabla J_n v_n\big) d\tau := F_n(v_n)(t)$$

- For $T > 0$ that will be fixed later we introduce

$$B_T = \Big\{ v \in L^\infty([0,T]; H^s); \|v\|_{T,s} \leq 2\|v_0\|_{H^s} \Big\},$$

with $\|v\|_{L^\infty_T H^s} := \|v\|_{T,s}$.

- Claim: For small T, $F_n : B_T \to B_T$ is a contraction.

1. Stability: let $v \in B_T$

$$\begin{aligned} \|F_n(v)\|_{T,s} &\leq \|v_0\|_s + CT2^n\|v\|^2_{T,s} \\ &\leq \|v_0\|_s + 4CT2^n\|v_0\|^2_s \leq 2\|v_0\|_s, \end{aligned}$$

provided $4CT2^n\|v_0\|_s \leq 1$.

2. Contraction: let $v, w \in B_T$

$$\begin{aligned} \|F(v) - F(w)\|_{T,s} &\leq CT2^n\|v - w\|_{T,s}\Big(\|v\|_{T,s} + \|w\|_{T,s}\Big) \\ &\leq 4CT2^n\|v_0\|_s\|v - w\|_{T,s} \end{aligned}$$

- Therefore, there exist $T_n > 0$ such that (E_n) admit a unique solution $v_n \in B_T$.
- Claim: $T_n^\star = +\infty$. Indeed,
  1. From the law product

$$\begin{aligned}\|v_n(t)\|_s &\leq \|v_0\|_s + C2^n \int_0^t \|J_n v_n(\tau)\|_s \|v_n(\tau)\|_s d\tau \\ &\leq \|v_0\|_s + C2^{n(1+s)} \int_0^t \|v_n(\tau)_{L^2} \|v_n(\tau)\|_s d\tau\end{aligned}$$

  2. Conservation of the energy: since $\operatorname{div} v_n = 0$ then

$$\|v_n(t)\|_{L^2} = \|J_n v_0\|_{L^2}$$

  3. We conclude by using Gronwall inequality.

Uniform e

- From the uniquene

$J_n v_n = v_n$ and

$$\frac{d}{dt} v_n = -J_n \mathbb{P}\big(v_n \cdot \nabla v_n\big)$$

- Localizing in frequency

$$\frac{d}{dt}\Delta_q v_n + J_n \mathbb{P}\big(v_n \cdot \nabla \Delta_q v_n\big) = -J_n \mathbb{P}\big\{\big[\Delta_q, v_n\big] \cdot \nabla v_n\big\}$$

- We conclude as for the a priori e

$$\|v_n(t)\|_{H^s} \leq \|v_0\|_{H^s} + C \int_0^t \|\nabla v_n(\tau)_{L^\infty} \|v_n(\tau)\|_{H^s} d\tau$$

- Thus there exist $T \geq \frac{C}{\|v_0\|_s}$ such that

$$\|v_n\|_{T,s} \leq 2\|v_0\|_s$$

Strong convergence

- $\{v_n\}$ is a Cauchy sequence in $L^\infty([0,T];L^2)$
  1. Let $n \geq m$ and $w_{n,m} = v_n - v_m$, then

$$\begin{aligned}\frac{d}{dt} w_{n,m} &= -\mathbb{P}J_n\big(w_{n,m}\cdot\nabla v_n\big) + \mathbb{P}J_n\big(v_m\cdot\nabla w_{n,m}\big)\\ &+ \mathbb{P}(J_n - J_m)\big(v_m\cdot\nabla v_m\big)\end{aligned}$$

  2. Energy e

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_{n,m}(t)\|_{L^2}^2 &\leq C\|\nabla v_n(t)\|_{L^\infty}\|w_{n,m}(t)\|_{L^2}^2\\ &+ 2^{-m}\|v_m\cdot\nabla v_m(t)\|_{H^1}\|w_{n,m}(t)\|_{L^2}\\ &\leq C\|v_0\|_{H^s}\|w_{n,m}(t)\|_{L^2}^2\\ &+ C2^{-m}\|v_0\|_{H^s}^2\|w_{n,m}(t)\|_{L^2}\end{aligned}$$

- By interpolation $\{v_n\}$ is a Cauchy sequence in $L^\infty([0,T];H^{s'}), \quad \forall s' < s$
- Therefore there exist $v \in L^\infty([0,T];H^s)$ such that

$$(v_n) \to v \text{ in } L^\infty([0,T];H^{s'})$$

- This allow (E_n)

  and prove the existence of a solution for Euler equations.

Continuity in time

- The fact that $v \in \mathscr{C}([0,T]; H^s)$ follow

$$\sum_{q\geq -1} 2^{2qs} \|\Delta_q v\|^2_{L^\infty_T L^2} < \infty.$$

Indeed, with the notation $S_n = \sum_{q=-1}^{n-1} \Delta_q$

$$\begin{aligned} \|v(t) - v(\tau)\|_s &\leq |t-\tau| \|\partial_t S_n v\|_{T,s} + \Big(\sum_{q\geq n} 2^{2qs} \|\Delta_q v\|^2_{L^\infty_T L^2}\Big)^{\frac{1}{2}} \\ &\leq C\|v\|^2_{T,s} 2^n |t-\tau| + \varepsilon_n \end{aligned}$$

Beale-Kato-Majda criterion:

Let $v_0 \in H^s, s > \frac{d}{2} + 1$ and $v \in \mathscr{C}([0, T^\star[; H^s)$ be the solution to Euler sy

$$T^* < +\infty \Longrightarrow \int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = +\infty.$$

We recall that the vorticity ω is defined by

$$\omega = \mathrm{curl}\, u := \begin{cases} \partial_1 u^2 - \partial_2 u^1, & \text{if } d = 2, \\ \nabla \wedge u, & \text{if } d = 3. \end{cases}$$

- This is weaker than the Lipschitz condition on v:

$$\|\omega\|_{L^\infty} \le \|\nabla v\|_{L^\infty} \not\lesssim \|\omega\|_{L^\infty}$$

Proof

- Logarithmic e

  Let $s > \frac{d}{2} + 1$

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + (1 + \|\omega\|_{L^\infty}) \log(e + \|v\|_{H^s})$$

- Energy e

$$\|v(t)\|_{H^s} \leq \|v_0\|_{H^s} e^{CV(t)}, \quad V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

- Therefore

$$\|\nabla v(t)\|_{L^\infty} \leq C\|\omega(t)\|_{L^\infty} V(t) + l.o.t$$

- Integrating in time

$$V(t) \leq C \int_0^t \|\omega(\tau)\|_{L^\infty} V(\tau) d\tau + l.o.t$$

Global existence in $2d$

Let $v_0 \in H^s(\mathbb{R}^2), s > 2$ and $v \in C([0, T^\star[, H^s$ be the solution to Euler sy

$$T^* = +\infty$$

- This follow

$$\partial_t \omega + v \cdot \nabla \omega = 0$$

- Thus

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$$

- We conclude by B-K-M criterion.

Perturbation of order zero

- Consider the equation in $2d$

$$\partial_t v + v \cdot \nabla v + \nabla P = (v^1, 0), \quad \mathrm{div}\, v = 0$$

- We can construct local solutions in $H^s, s > 2$.
- Global existence is o

$$\partial_t \omega + v \cdot \nabla \omega = -\partial_2 v^1 = \partial_{22} \Delta^{-1} \omega$$

♦ We say that a vector-field $v = (v^1, v^2, v^3)$ is axisymmetric without swirl if

$$v(x, y, z) = v^r(r, z)\mathbf{e_r} + v^z(r, z)\mathbf{e_z},$$

whith $r = (x^2 + y^2)^{\frac{1}{2}}$, $\big(e_r, e_\theta, e_z\big)$ the cylindrical basis.

♦ Structure of the vorticity :

$$\omega = (\partial_z v^r - \partial_r v^z)e_\theta.$$

♦ By continuity we have $\omega(t, 0, z) = 0$, $v^r(t, 0, z) = 0$.

Sub-critical regularitie

1. Ukhov $v_0 \in H^s$, with $s > \frac{7}{2}$. Then the sy solution $\mathscr{C}(\mathbb{R}_+; H^s)$.
2. Shirota and Yanagisawa (1994): $T^* = +\infty$, if $v_0 \in H^s$, $s > 5/2$

Dynamic

- The vorticity satisfie

$$(\partial_t + v\cdot\nabla)\omega = \omega\cdot\nabla\, v.$$

- Moreover, the stretching term take

$$\omega\cdot\nabla\, v = v^r\frac{\omega}{r}\cdot$$

- It follow

$$(\partial_t + v^r\partial_r + v^z\partial_z)\omega = v^r\frac{\omega}{r}\cdot$$

- An easy computation give

$$(\partial_t + v\cdot\nabla)\frac{\omega}{r} = 0\cdot$$

Conservation law

- Thus we get new conservation law $p \in [1, \infty]$,

$$\|\omega(t)/r\|_{L^p} = \|\omega_0/r\|_{L^p}.$$

- B-K-M criterion: it suffice

$$\begin{aligned} \|\omega(t)\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty} + \int_0^t \|v\|_{L^\infty} \|\frac{\omega}{r}\|_{L^\infty} d\tau \\ &\lesssim 1 + \|v\|_{L^1_t L^\infty}. \end{aligned}$$

- Now we use

$$\begin{aligned}\|v(t)\|_{L^\infty} &\lesssim \|v(t)\|_{L^2} + \|\omega(t)\|_{L^\infty} \\ &\le \|v_0\|_{L^2} + \|\omega(t)\|_{L^\infty}\end{aligned}$$

- This give

$$\|\omega(t)\|_{L^\infty} \lesssim 1 + t + \int_0^t \|\omega(\tau)\|_{L^\infty} d\tau.$$

- We conclude by Gronwall inequality.

Critical regularities

Critical space C^1.

Example $H^{\frac{d}{2}+1}$, C^1, $B^{\frac{d}{2}+1}_{2,1}$, $B^1_{\infty,1}$

- Ill-po
  Elgindy-Masmoudi,..
- Local well-po
  Chae, Pak-Park,..
- What about global existence in $2d$ with critical regularitie

This is not given by B-K-M criterion.

Recall the vorticity-velocity formulation

$$\partial_t \omega + v \cdot \nabla \omega = 0, \quad \Delta v = \nabla^{\perp} \omega$$

- We shall discuss the global regularity for two space
  1. Beirao da Veiga 1984:

$$\omega_0 \in C_\star(\mathbb{R}^2), \quad \|f\|_\star = \|f\|_{L^\infty} + \int_0^1 \mu_f(r) \frac{dr}{r} < \infty$$

     with $\mu_f(r) := \sup_{|x-y| \leq r} |f(x) - f(y)|$
  2. Vishik 1998:

$$v_0 \in B^1_{\infty,1} \Longrightarrow \omega_0 \in B^0_{\infty,1}$$

$$\|f\|_{B^0_{\infty,1}} = \sum_{q \geq -1} \|\Delta_q f\|_{L^\infty}$$

- **Embeddings**

$$\forall s > 0, \quad C^s \hookrightarrow C_\star \hookrightarrow B^0_{\infty,1} \hookrightarrow C_b$$

[Beirao da Veiga] Let $\omega_0 \in C_\star \cap L^p, p \in]1,2[$ then Euler equations admit a unique global solution

$$\omega \in \mathscr{C}(\mathbb{R}_+; C_\star \cap L^p), \quad v \in \mathscr{C}(\mathbb{R}_+; C^1_b)$$

- Claim 1: for any $p < \infty$ there exist $C > 0$ such that

$$\|\nabla v\|_{L^\infty} \le C\|\omega\|_{L^p} + C\|\omega\|_\star$$

This follow

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy$$

- Thus

$$
\begin{aligned}
|\nabla v(x)| &\leq C \int_{|x-y|\geq 1} \frac{1}{|x-y|^2} |\omega(y)| dy \\
&+ C \int_{|x-y|\leq 1} \frac{1}{|x-y|^2} |\omega(y) - \omega(x))| dy \\
&\leq C\|\omega\|_{L^p} + C \int_0^1 \mu_\omega(r) \frac{dr}{r}
\end{aligned}
$$

- Claim 2:

$$
\|\omega(t)\|_\star \leq C\|\omega_0\|_\star \Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big)
$$

This is based on the control of

$$\mu_{\omega_t}(r) = \sup_{|x-y|\le r} |\omega_0(\psi^{-1}(t,x)) - \omega_0(\psi^{-1}(t,y))|$$

- First we have

$$\Big|(\psi^{-1}(t,x) - \psi^{-1}(t,y))\Big| \le |x-y|\, e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}$$

- Consequently $\mu_{\omega_t}\Big(r\, e^{-CV(t)}\Big) \le \mu_{\omega_0}(r).$
- This implie

$$\begin{aligned}\|\omega_t\|_\star \quad &\le \quad \|\omega_0\|_{L^\infty} + \int_0^1 \mu_{\omega_t}\Big(r\, e^{-CV(t)}\Big)\frac{dr}{r} \\ &+ \quad \int_{e^{-CV(t)}}^1 \mu_{\omega_t}\big(r\big)\frac{dr}{r}\end{aligned}$$

- Hence we get

$$\begin{aligned} \|\omega_t\|_\star &\leq \|\omega_0\|_\star + 2\|\omega_0\|_{L^\infty} \int_{e^{-CV(t)}}^{1} \frac{dr}{r} \\ &\leq \|\omega_0\|_\star + C\|\omega_0\|_{L^\infty} V(t) \end{aligned}$$

- We conclude from Claim1 and Claim2 that

$$\forall t \geq 0, \quad \|\omega_t\|_\star + \|\nabla v(t)\|_{L^\infty} \leq Ce^{Ct}$$

[Vishik] Let $v_0 \in B^1_{\infty,1}$ then Euler equations admit a unique solution

$$v \in \mathscr{C}(\mathbb{R}_+; B^1_{\infty,1}).$$

- First note that

$$\|v(t)\|_{B^1_{\infty,1}} \leq \|v(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}$$

- The e $\|v(t)\|_{L^\infty}$ is done by Serfati.
- The e $\|\omega(t)\|_{B^0_{\infty,1}}$ use

  e

$\mathcal{E}$ $\|v(t)\|_{L^\infty}$

- Let $N \in \mathbb{N}$ and denote by S_{-N} the o

$$S_{-N} = \chi(2^N\sqrt{-\Delta}), \quad \chi \equiv 1 \text{ around } 0.$$

- Then

$$\begin{aligned}
\|v(t)\|_{L^\infty} &\leq \|S_{-N}v(t)\|_{L^\infty} + \|(\mathrm{Id} - S_{-N})v(t)\|_{L^\infty} \\
&\leq \|S_{-N}v(t)\|_{L^\infty} + \|(\mathrm{Id} - S_{-N})\Delta^{-1}\nabla^\perp\omega(t)\|_{L^\infty} \\
&\leq \|S_{-N}v(t)\|_{L^\infty} + C2^N\|\omega(t)\|_{L^\infty} \\
&\leq \|S_{-N}v(t)\|_{L^\infty} + C2^N\|\omega_0\|_{L^\infty}
\end{aligned}$$

- On the other hand

$$
\begin{aligned}
S_{-N}v(t) &= S_{-N}v_0 - \int_0^t S_{-N}\mathbb{P}(v\cdot\nabla v)(\tau)d\tau \\
&= S_{-N}v_0 - \int_0^t S_{-N}\mathbb{P}\mathrm{div}\,(v\otimes v)(\tau)d\tau
\end{aligned}
$$

- Therefore

$$
\begin{aligned}
\|S_{-N}v(t)\|_{L^\infty} &\leq C\|v_0\|_{L^\infty} + C2^{-N}\int_0^t \|(v\otimes v)(\tau)\|_{L^\infty}d\tau \\
&= C\|v_0\|_{L^\infty} + C2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau
\end{aligned}
$$

- It follow

$$\begin{aligned}\|v(t)\|_{L^\infty} &\leq C\|v_0\|_{L^\infty} + C2^N\|\omega_0\|_{L^\infty} + C2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau \\ &\leq C\|v_0\|_{L^\infty} + C\|\omega_0\|_{L^\infty}^{\frac{1}{2}}\Big(\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau\Big)^{\frac{1}{2}}\end{aligned}$$

- We apply Gronwall inequality

$$\|v(t)\|_{L^\infty} \leq C\|v_0\|_{L^\infty} e^{Ct\|\omega_0\|_{L^\infty}}$$

$\mathcal{E}$ $\|\omega(t)\|_{B^0_{\infty,1}}$

Recall that $\|f\|_{B^0_{\infty,1}} = \sum_{q\geq -1} \|\Delta_q f\|_{L^\infty}$ and

$$\left\{f;\ \ \widehat{f} \in L^1\right\} \hookrightarrow B^0_{\infty,1} \hookrightarrow C_b$$

Let ψ be a volume-pre

$f \in B^0_{\infty,1}$. Then $f \circ \psi \in B^0_{\infty,1}$ and

$$\|f \circ \psi\|_{B^0_{\infty,1}} \leq C\|f\|_{B^0_{\infty,1}} \log\Big(\|\nabla\psi\|_{L^\infty} + (\|\nabla\psi^{-1}\|_{L^\infty}\Big).$$

- We write

$$\omega(t,x) = \omega_0(\psi^{-1}(t,x)),$$

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau,x))d\tau$$

- Since

$$\|\nabla\psi^{\pm 1}(t)\|_{L^\infty} \leq Ce^{CV(t)}, \quad V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

- Therefore

$$\begin{aligned} \|\omega(t)\|_{B^0_{\infty,1}} &\leq C\|\omega_0\|_{B^1_{\infty,1}}\Big(1 + V(t)\Big) \\ &\leq C\|\omega_0\|_{B^1_{\infty,1}}\Big(1 + \int_0^t \|v(\tau)\|_{L^\infty} d\tau + \int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}} d\tau\Big) \end{aligned}$$

Extension to transport-diffusion model

- We consider the equation:

$$\begin{cases} \partial_t f + v \cdot \nabla f - \nu \Delta f = 0, & \nu \geq 0 \\ f_{|t=0} = f_0 \in B^0_{\infty,1}. \end{cases}$$

- We shall study the persistence of the initial regularity uniformly on $\nu \geq 0$.

[**H.-Keraani** 2008] There exist $C > 0$ inde ν such that,

$$\|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right)$$

The proof is based on the following decompo

$$f(t,x) = \sum_{q\geq -1} \widetilde{f}_q(t,x) \quad \text{with}$$

$$\begin{cases} \partial_t \widetilde{f}_q + v\cdot\nabla\widetilde{f}_q - \nu\Delta\widetilde{f}_q = 0, \\ \widetilde{f}_{q|t=0} = \Delta_q f_0. \end{cases}$$

1. Maximum principle: $\|\widetilde{f}_q(t)\|_{L^\infty} \leq \|\Delta_q f_0\|_{L^\infty}$.
2. Frequency decay: For every $j,q \in \mathbb{N}\cup\{-1\}$ we have

$$\|\Delta_j \widetilde{f}_q(t)\|_{L^\infty} \leq C2^{-\frac{1}{2}|j-q|}\|\Delta_q f_0\|_{L^\infty} e^{CV(t)}.$$

- Now let $N \in \mathbb{N}$ a free number

$$
\begin{aligned}
\|f(t)\|_{B^0_{\infty,1}} &\leq \sum_{|j-q|\leq N} \|\Delta_j \widetilde{f_q}(t)\|_{L^\infty} + \sum_{|j-q|>N} \|\Delta_j \widetilde{f_q}(t)\|_{L^\infty} \\
&\leq C \sum_{|j-q|\leq N} \|\Delta_q f_0\|_{L^\infty} + Ce^{CV(t)} \sum_{|j-q|>N} 2^{-\frac{1}{2}|j-q|} \|\Delta_q f_0\|_{L^\infty} \\
&\leq CN\|f_0\|_{B^0_{\infty,1}} + C2^{-\frac{1}{2}N} e^{CV(t)} \|f_0\|_{B^0_{\infty,1}}.
\end{aligned}
$$

- We choo $N \approx V(t)$ and thus

$$
\|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \big(1 + V(t)\big)
$$

Axisymmetric case: critical space

1. Danchin (2006): $T^* = +\infty$ for

$$\omega_0 \in L^\infty \cap L^{3,1} \text{ and } \omega_0/r \in L^{3,1}.$$

2. Abidi-H.-Keraani 2010: $T^* = +\infty$ for

$$1)\ v_0 \in B_{p,1}^{\frac{3}{p}+1}, \quad 2)\ \frac{\omega_0}{r} \in L^{3,1}.$$

Thank You!