Analyse mathématique d'un modèle de fluide radiatif

Raphaël Danchin, Université Paris-Est Créteil, France
Bernard Ducomet, CEA, France

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- **The barotropic Navier-Stokes equations:**

$$\begin{cases} \partial_t \varrho + \operatorname{div}_x (\varrho \vec{u}) = 0, \\ \partial_t (\varrho \vec{u}) + \operatorname{div}_x (\varrho \vec{u} \otimes \vec{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \operatorname{div}_x \mathbb{T} - \vec{S}_F, \end{cases} \qquad (t,x) \in (0,T) \times \mathbb{R}^n.$$

where the pressure law is $p = P(\varrho)$ with P a smooth given function and $\mathbb{T}$ stands for the **viscous stress tensor** given by

$$\mathbb{T} = \mu \left(\nabla_x \vec{u} + \nabla_x^t \vec{u} \right) + \lambda \ \operatorname{div}_x \vec{u} \, \mathbb{I}_n, \qquad \mu > 0 \ \text{ and } \ \lambda + 2\mu > 0.$$

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- **Radiation force :** $\vec{S}_F := \int_{\mathcal{S}^{n-1}} \int_0^\infty \vec{\omega} S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}.$
- The **radiative source** S satisfies $S = S_{a,e} + S_s$ with

$$S_{a,e} = \mathcal{L} \sigma_a \Big(B(\nu, \varrho) - I \Big) \ \text{ and } \ S_s = \mathcal{L} \mathcal{L}_s \sigma_s \left(\frac{1}{|\mathcal{S}^{n-1}|} \int_{\mathcal{S}^{n-1}} I(\cdot, \vec{\omega}) \, \mathrm{d}\vec{\omega} - I \right) \cdot$$

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- Ma and Re designate the Mach and Reynold numbers, respectively.
- $\mathcal{C}$ is the ratio of the light velocity over the reference velocity ($\mathcal{C} \gg 1$).
- $\mathcal{L}$ and $\mathcal{L}_s$ measure the influence of radiation.

Simplifying assumptions

- **Isotropy :** σ_a and σ_s are independent of $\vec{\omega}$.
- **"Grey" case :** σ_a, σ_s and the distribution function B are independent of ν.

$\rightsquigarrow$ the unknowns depend only on t, x and $\vec{\omega}$.

The averaged radiative source s and radiative momentum $\vec{s}_F$ are given by:

$$s := \mathcal{L}\sigma_a(B - I) + \mathcal{L}\sigma_s(< I > - I) \quad \text{with} \quad < I > := \int_{\mathcal{S}^{n-1}} I(t, x, \vec{\omega})\, \mathrm{d}\vec{\omega},$$

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- **P1 approximation :** Postulate that

$$I = I_0 + \vec{I}_1 \cdot \vec{\omega} \quad \text{with} \quad I_0 = I_0(t, x) \quad \text{and} \quad \vec{I}_1 = \vec{I}_1(t, x).$$

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$\rightsquigarrow$ Plugging that ansatz in the radiative transfer equation yields :

$$\begin{cases} \frac{1}{c}\,\partial_t I_0 + \frac{1}{n}\ \mathrm{div}_x\,\vec{I}_1 = \mathcal{L}\sigma_a(B(\varrho) - I_0), \\ \frac{1}{c}\,\partial_t \vec{I}_1 + \nabla_x I_0 = -\mathcal{L}(\sigma_a + \sigma_s)\vec{I}_1. \end{cases}$$

The radiative force becomes

$$\vec{S}_F = -\left(\frac{\mathcal{L}(\sigma_a + \mathcal{L}_s\sigma_s)}{n}\right)\vec{I}_1.$$

Final shape of the system

Reference constant equilibrium : $\varrho = 1, \vec{u} = \vec{0}, I_0 = B(1), \vec{I}_1 = \vec{0}.$

The system for $a := \varrho - 1$, $\vec{u}$, $j_0 = I_0 - B(1)$ and $\vec{j}_1 = \vec{I}_1$ reads

$$\begin{cases} \partial_t a + \vec{u} \cdot \nabla a = -(1+a)\mathrm{div}\, \vec{u}, \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} - \frac{1}{\mathrm{Re}} \frac{1}{1+a} \mathcal{A}\vec{u} + \frac{1}{\mathrm{Ma}^2} \nabla(\Pi(1+a)) = \frac{1}{n} \frac{1}{1+a} \mathcal{L}(\sigma_a + \mathcal{L}_s \sigma_s)\vec{j}_1, \\ \partial_t j_0 + \frac{1}{n} \mathcal{C}\mathrm{div}\, \vec{j}_1 = \mathcal{C}\mathcal{L}\sigma_a(B(1+a) - B(1) - j_0), \\ \partial_t \vec{j}_1 + \mathcal{C}\nabla j_0 = -\mathcal{C}\mathcal{L}(\sigma_a + \mathcal{L}_s \sigma_s)\vec{j}_1, \end{cases}$$

with $\mathcal{A} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ and $\Pi(\varrho) := \int_1^{\varrho} \frac{P'(s)}{s}\, ds.$

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We get the following system for $(b, \vec{u}, j_0, \vec{j}_1)$:

$$\begin{cases} \partial_t b + \vec{u} \cdot \nabla b + (1 + k_1(b)) \mathrm{div}\, \vec{u} = 0, \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} - \frac{1}{\mathrm{Re}}(1+k_2(b)) \mathcal{A} \vec{u} + \frac{\alpha}{\mathrm{Ma}^2}(1+k_3(b)) \nabla b = \frac{\mathcal{L}(\sigma_a + \mathcal{L}_s \sigma_s)}{n}(1+k_4(b)) \vec{j}_1 \\ \partial_t j_0 + \frac{1}{n}\, \mathcal{C} \mathrm{div}\, \vec{j}_1 = \mathcal{C} \mathcal{L} \sigma_a (\underline{\alpha}'\, b - j_0), \\ \partial_t \vec{j}_1 + \mathcal{C} \nabla j_0 = -\mathcal{C} \mathcal{L} (\sigma_a + \mathcal{L}_s \sigma_s) \vec{j}_1, \end{cases}$$

with $\underline{\alpha} := P'(1)$, $\underline{\alpha}' := B'(1)$ and $k_1,\ k_2,\ k_3,\ k_4$ vanishing at 0.

Linearized equations about $b=0,\ \vec{u}=\vec{0},\ j_0=0,\ \vec{j}_1=\vec{0}$:

$$\begin{cases} \partial_t b + \operatorname{div}\vec{u} = 0, \\ \partial_t \vec{u} - \frac{1}{\mathrm{Re}}\ \mu\Delta\vec{u} - \frac{1}{\mathrm{Re}}(\lambda+\mu)\nabla\operatorname{div}\vec{u} + \frac{1}{\mathrm{Ma}^2}\ \underline{\alpha}\nabla b = n^{-1}\mathcal{L}(\sigma_a + \mathcal{L}_s\sigma_s)\vec{j}_1, \\ \partial_t j_0 + \frac{1}{n}\ \mathcal{C}\operatorname{div}\vec{j}_1 = \mathcal{C}\mathcal{L}\sigma_a(\underline{\alpha}'b - j_0), \\ \partial_t\vec{j}_1 + \mathcal{C}\nabla j_0 = -\mathcal{C}\mathcal{L}(\sigma_a+\sigma_s)\vec{j}_1. \end{cases}$$

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Weak coupling for the incompressible modes $\mathcal{P}\vec{u}$ **and** $\mathcal{P}\vec{j}_1$:

$$\partial_t\mathcal{P}\vec{u}-\frac{\mu}{\mathrm{Re}}\Delta\mathcal{P}\vec{u}=\mathcal{L}\left(\frac{\sigma_a+\mathcal{L}_s\sigma_s}{n}\right)\mathcal{P}\vec{j}_1 \quad \text{and} \quad \partial_t\mathcal{P}\vec{j}_1+\mathcal{C}\mathcal{L}(\sigma_a+\mathcal{L}_s\sigma_s)\mathcal{P}\vec{j}_1=\vec{0}.$$

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Weak coupling for the incompressible modes $\mathcal{P}\vec u$ **and** $\mathcal{P}\vec j_1$:

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Compressible unknowns : $d:=\Lambda^{-1}\operatorname{div}\vec u$ and $j_1:=\Lambda^{-1}\operatorname{div}\vec j_1$.

The 4×4 system for (b,d,j_0,j_1) reads

$$\begin{cases} \partial_t b+\Lambda d=0,\\ \partial_t d-\frac{1}{\mathrm{Re}}\,\nu\Delta d-\frac{1}{\mathrm{Ma}^2}\,\underline{\alpha}\Lambda b=n^{-1}\mathcal{L}(\sigma_a+\mathcal{L}_s\sigma_s)j_1,\\ \partial_t j_0+\frac{1}{n}\,\mathcal{C}\Lambda+j_1\mathcal{C}\mathcal{L}\sigma_a j_0=\mathcal{C}\mathcal{L}\sigma_a\underline{\alpha}'b,\\ \partial_t j_1-\mathcal{C}\Lambda j_0+\mathcal{C}\mathcal{L}(\sigma_a+\mathcal{L}_s\sigma_s)j_1=0. \end{cases}$$

Reduction of number of parameters

One may reduce the number of parameters to 'only' 5 by rescaling:

$$b(t,x)=\widetilde{b}(\tau t,\chi x),\ d(t,x)=\delta\widetilde{d}(\tau t,\chi x),\ j_0(t,x)=\zeta_0\widetilde{j}_0(\tau t,\chi x),\ j_1(t,x)=\zeta_1\widetilde{j}_1(\tau t,\chi x)$$

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In Fourier variables ξ, setting $\rho:=|\xi|$, we get

$$\frac{d}{dt}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix}+\underbrace{\begin{pmatrix}0&\rho&0&0\\ -\rho&\rho^2&0&-\varsigma\\ -\varsigma&0&\beta&\alpha\rho\\ 0&0&-\alpha\rho&\gamma\end{pmatrix}}_{A_\rho}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix}=0$$

with

$$\alpha=\frac{1}{\sqrt{n}}\mathcal{C}\frac{\mathrm{Ma}}{\sqrt{\underline{\alpha}}},\qquad \beta=\sigma_a\mathcal{C}\mathcal{L}\frac{\nu}{\mathrm{Re}}\frac{\mathrm{Ma}^2}{\underline{\alpha}},$$

$$\gamma=\left(1+\frac{\sigma_a}{\sigma_s}\right)\left(\frac{\sigma_a+\mathcal{L}_s\sigma_s}{n}\right),\qquad \varsigma=\frac{1}{n^{1/4}}\frac{\nu}{\mathrm{Re}}\left(\frac{\mathrm{Ma}^2}{\underline{\alpha}}\right)^{\frac{5}{4}}\mathcal{L}\sqrt{\underline{\alpha}'\mathcal{C}\sigma_a(\sigma_a+\mathcal{L}_s\sigma_s)}.$$

High frequency analysis

Heuristics : *Radiative coupling is low order for large ρ :*

$$\begin{cases} \partial_t\widehat{b}+\rho\widehat{d}=0,\\ \partial_t\widehat{d}-\rho\widehat{b}+\rho^2\widehat{d}=\varsigma\widehat{j}_1,\\ \partial_t\widehat{j}_0+\beta\widehat{j}_0+\alpha\rho\widehat{j}_1=\varsigma\widehat{b},\\ \partial_t\widehat{j}_1+\gamma\widehat{j}_1-\alpha\rho\widehat{j}_0=0.\end{cases}$$

Lyapunov functional:

$$\mathcal{L}_\rho^2:=|\rho\widehat{b}|^2-2\rho\,\mathrm{Re}\,(\widehat{b}\overline{\widehat{d}})+2(|\widehat{b}|^2+|\widehat{d}|^2)+\frac{\varsigma^2}{\gamma}(|\widehat{j}_0|^2+|\widehat{j}_1|^2)$$

satisfies $\mathcal{L}_\rho^2(t)\leq e^{-\min(\beta,\gamma)t}\mathcal{L}_\rho^2(0)$ and $\mathcal{L}_\rho\approx|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)|$ for $\rho\geq\rho_h$.

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satisfies $\mathcal{L}_\rho^2(t) \leq e^{-\min(\beta,\gamma)t}\mathcal{L}_\rho^2(0)$ and $\mathcal{L}_\rho \approx |(\rho\widehat{b}, \widehat{d}, \widehat{j}_0, \widehat{j}_1)|$ for $\rho \geq \rho_h$.

Plugging that information in the equation for $\widehat{d}$:

$$\partial_t\widehat{d} + \rho^2\widehat{d} = \varsigma\widehat{j}_1 + \rho\widehat{b},$$

we get the following 'parabolic' gain of regularity:

$$\rho^2\int_0^t |\widehat{d}|\,d\tau \leq |\widehat{d}(0)| + \frac{2}{\min(\beta,\gamma)}\left(1 + \frac{2\gamma}{\varsigma}\right)\left(\rho|\widehat{b}(0)| + |\widehat{d}(0)| + \frac{\varsigma^2}{\gamma}\left(|\widehat{j}_0(0)| + |\widehat{j}_1(0)|\right)\right).$$

The low frequency analysis (small ρ)

Setting $U = QX$ (corresponding to diagonalization for $\rho = 0$) gives:

$$(E) \qquad \partial_t U + A_0 U + \rho\,(A_1 + B_1)\,U + \rho^2 A_2 U = 0.$$

- A_0 is diagonal nonnegative (but degenerate).
- A_1 is skew-symmetric (or almost).
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$\rightsquigarrow$ Change of unknown $V = (\mathrm{Id} + \rho P)U$ so as **to kill the term with** B_1.

Whenever $(\mathrm{Id} + \rho P)$ is invertible, the equation for V reads :

$$\partial_t V + A_0 V + \rho\big(A_1 + B_1 + [P, A_0]\big) + \rho^2\big([A_0, P]P + [P, A_1] + [P, B_1] + A_2\big)V = \mathcal{O}(\rho^3).$$

Therefore, if one can find some matrix P so that $[A_0, P] = B_1$ then

$$\partial_t V + A_0 V + \rho A_1 V + \rho^2\,(A_2 + PB_1 + [P, A_1])\,V = \mathcal{O}(\rho^3).$$

We rewrite all our matrices in block form (each block having size 2×2):

$$B_1=\begin{pmatrix} 0 & B_1^1 \\ B_1^2 & 0\end{pmatrix},\quad A_0=\begin{pmatrix} 0 & 0 \\ 0 & \Delta\end{pmatrix},\quad P=\begin{pmatrix} P^{11} & P^{12} \\ P^{21} & P^{22}\end{pmatrix}.$$

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Computing the commutator

$$[A_0, P] = \begin{pmatrix} 0 & -P^{12}\Delta \\ \Delta P^{21} & [\Delta, P^{22}] \end{pmatrix},$$

we see that $[A_0, P] = B_1$ if, for example,

$$P^{11} := 0, \quad P^{22} := 0, \quad P^{12} := -B_1^1 \Delta^{-1}, \quad P^{21} := \Delta^{-1} B_1^2.$$

We rewrite all our matrices in block form (each block having size 2×2):

$$B_1 = \begin{pmatrix} 0 & B_1^1 \\ B_1^2 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}, \quad P = \begin{pmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{pmatrix}.$$

Computing the commutator

$$[A_0, P] = \begin{pmatrix} 0 & -P^{12}\Delta \\ \Delta P^{21} & [\Delta, P^{22}] \end{pmatrix},$$

we see that $[A_0, P] = B_1$ if, for example,

$$P^{11} := 0, \quad P^{22} := 0, \quad P^{12} := -B_1^1 \Delta^{-1}, \quad P^{21} := \Delta^{-1} B_1^2.$$

In other words,

$$P = \begin{pmatrix} 0 & 0 & 0 & \frac{\varsigma}{\gamma^2} \\ 0 & 0 & \frac{\alpha\varsigma}{\beta\gamma} & 0 \\ 0 & -\frac{\varsigma}{\beta^2} & 0 & 0 \\ -\frac{\alpha\varsigma}{\beta\gamma} & 0 & 0 & 0 \end{pmatrix},$$

which eventually leads to the following change of unknowns :

$$V = \begin{pmatrix} \widehat{\mathfrak{b}} \\ \widehat{\mathfrak{d}} \\ \widehat{\mathfrak{j}}_0 \\ \widehat{\mathfrak{j}}_1 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & \frac{\varsigma}{\gamma^2}\rho \\ -\frac{\alpha\varsigma^2}{\beta^2\gamma}\rho & 1 & \frac{\alpha\varsigma}{\beta\gamma}\rho & \frac{\varsigma}{\gamma} \\ -\frac{\varsigma}{\beta} & -\frac{\varsigma}{\beta^2}\rho & 1 & -\frac{\varsigma^2}{\beta^2\gamma}\rho \\ -\frac{\alpha\varsigma}{\beta\gamma}\rho & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{b} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix}.$$

Parabolic unknowns $(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})$**:** up to an $\mathcal{O}(\rho^3)$ term, we have

$$\begin{cases} \partial_t\widehat{\mathfrak{b}}+\rho\widehat{\mathfrak{d}}-\rho^2\frac{\alpha\varsigma^2}{\beta\gamma^2}\widehat{\mathfrak{b}}=\kappa_0\rho^2\widehat{\mathfrak{j}_0}, \\ \partial_t\widehat{\mathfrak{d}}-\rho\big(1+\frac{\alpha\varsigma^2}{\beta\gamma}\big)\widehat{\mathfrak{b}}+\rho^2\big(1-\frac{\alpha\varsigma^2}{\beta^2\gamma}\big)\widehat{\mathfrak{d}}=\kappa_1\rho^2\widehat{\mathfrak{j}_1}. \end{cases}$$

Lyapunov functional : $\mathcal{L}_\rho^2=\gamma\Big(1+\frac{\alpha\varsigma^2}{\beta\gamma}\Big)|\widehat{\mathfrak{b}}|^2+|\widehat{\mathfrak{d}}|^2-\Big(1+\frac{\alpha\varsigma^2}{\beta\gamma}\big(\frac{1}{\gamma}-\frac{1}{\beta}\big)\Big)\rho\,\mathrm{Re}\,(\widehat{\mathfrak{b}}\overline{\widehat{\mathfrak{d}}}).$

Stability condition : $\frac{\alpha\varsigma^2}{\beta\gamma}\big(\frac{1}{\beta}+\frac{1}{\gamma}\big)<1.$

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Damped unknowns $(\widehat{\mathfrak{j}_0},\widehat{\mathfrak{j}_1})$**:** up to an $\mathcal{O}(\rho^3)$ term, we have

$$\begin{cases} \partial_t\widehat{\mathfrak{j}_0}+\big(\beta+\frac{\alpha\varsigma^2}{\beta^2\gamma}\rho^2\big)\widehat{\mathfrak{j}_0}+\widetilde{\alpha}\rho\mathfrak{j}_0=\rho^2\kappa_a\widehat{a}, \\ \partial_t\widehat{\mathfrak{j}_1}+\big(\gamma+\frac{\alpha\varsigma^2}{\beta\gamma^2}\rho^2\big)\widehat{\mathfrak{j}_1}-\alpha\rho\mathfrak{j}_1=\rho^2\kappa_d\widehat{d}. \end{cases}$$

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Final estimate for $\rho\leq\rho_\ell$:

$$|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau+\min(\beta,\gamma)\int_0^t|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau\lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|.$$

Medium frequency analysis

Wanted : exponential decay for any $\rho > 0$.

Let $s = \beta + \gamma$ and $p = \beta\gamma$. The characteristic polynomial of the matrix A_ρ reads

$$P_\rho(\lambda) = a_0\lambda^4 - a_1\lambda^3 + a_2\lambda^2 - a_3\lambda + a_4 \quad \text{with}$$

$$a_0 = 1, \quad a_1 = \rho^2 + s, \quad a_2 = (\alpha^2 + s + 1)\rho^2 + p,$$
$$a_3 = \alpha^2\rho^4 + (s+p)\rho^2, \quad a_4 = \alpha^2\rho^4 + (\varsigma^2\alpha + p)\rho^2.$$

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Routh-Hurwitz theorem: the roots of P_ρ have *positive* real part if and only if the following determinants are positive:

$$A_1 = a_1, \quad A_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix}, \quad A_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ 0 & a_4 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{vmatrix}.$$

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Those conditions are satisfied for all $\rho > 0$ if and only if

$$p^2 > \alpha\varsigma^2 s.$$

Medium frequency analysis

- We know that under the necessary and sufficient stability condition $p^2 > \alpha\varsigma^2 s$ then for all $\rho > 0$, we have $e^{-tA_\rho} \to 0$ when $t \to +\infty$.

- Fix some $0 < \underline{\rho} < \overline{\rho}$ and define, for $\rho \in [\rho_\ell, \rho_h]$,

$$T_\rho := \sup\{t \in \mathbb{R}^+ : |e^{-tA_\rho}| \geq e^{-1}\}.$$

- We know that $\rho \mapsto T_\rho$ is valued in $\mathbb{R}^+$ (stability condition). Besides, it is upper semi-continuous. Hence

$$T^* := \sup_{\rho \in [\rho_\ell, \rho_h]} T_\rho \quad \text{is finite.}$$

- This implies :

$$\exists c, C > 0,\ \forall \rho \in [\rho_\ell, \rho_h], \forall t \in \mathbb{R}^+,\ |e^{-tA_\rho}| \leq Ce^{-ct}.$$

A priori estimates for the linearized equations

Fourier localization of the linearized equations (Littlewood-Paley):

- **Low frequencies:** use

$$\mathcal{L}_k^2 := \left(1+\frac{\alpha\varsigma^2}{\beta\gamma}\right)\|\dot\Delta_k\mathfrak{b}\|_{L^2}^2 + \|\dot\Delta_k\vec{\mathfrak{u}}\|_{L^2}^2 + \left(1+\frac{\varsigma^2\alpha(\beta-\gamma)}{\beta^2\gamma^2}\right)\int\nabla\dot\Delta_k\mathfrak{b}\cdot\dot\Delta_k\vec{\mathfrak{u}}\,dx$$

$$\text{and}\quad \mathcal{J}_k^2 := \|\dot\Delta_k\mathfrak{j}_0\|_{L^2}^2 + \frac{\widetilde\alpha}{\alpha}\|\dot\Delta_k\vec{\mathfrak{j}}_1\|_{L^2}^2$$

with $\mathfrak{b} := b + \frac{\varsigma}{\gamma^2}\operatorname{div}\vec{j}_1,\ \vec{\mathfrak{u}} := \vec{u} + \frac{\varsigma}{\gamma}\vec{\mathcal{Q}}j_1 + \frac{\alpha\varsigma}{\beta\gamma}\nabla j_0 - \frac{\alpha\varsigma^2}{\beta^2\gamma}\nabla b,$
$\mathfrak{j}_0 := j_0 - \frac{\varsigma}{\beta}b - \frac{\varsigma}{\beta^2}\operatorname{div}\vec{u} - \frac{\varsigma^2}{\beta^2\gamma}\operatorname{div}\vec{j}_1$ and $\vec{\mathfrak{j}}_1 := \vec{j}_1 - \frac{\alpha\varsigma}{\beta\gamma}\nabla b,$

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- **High frequencies:** use

$$\mathcal{L}_k^2 = \|\nabla\dot\Delta_k b\|_{L^2}^2 + 2\int\nabla\dot\Delta_k b\cdot\dot\Delta_k\vec{u}\,dx + 2\|(\dot\Delta_k b,\dot\Delta_k\vec{u})\|_{L^2}^2 + \frac{\varsigma^2}{\gamma}\|(\dot\Delta_k j_0,\dot\Delta_k\vec{j}_1)\|_{L^2}^2.$$

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$$\text{and}\quad \mathcal{J}_k^2 := \|\dot\Delta_k\mathfrak{j}_0\|_{L^2}^2 + \frac{\widetilde{\alpha}}{\alpha}\|\dot\Delta_k\vec{\mathfrak{j}}_1\|_{L^2}^2$$

with $\mathfrak{b} := b + \frac{\varsigma}{\gamma^2}\operatorname{div}\vec{j}_1$, $\vec{\mathfrak{u}} := \vec{u} + \frac{\varsigma}{\gamma}\vec{\mathcal{Q}}j_1 + \frac{\alpha\varsigma}{\beta\gamma}\nabla j_0 - \frac{\alpha\varsigma^2}{\beta^2\gamma}\nabla b$,
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- **High frequencies:** use

$$\mathcal{L}_k^2 = \|\nabla\dot\Delta_k b\|_{L^2}^2 + 2\int\nabla\dot\Delta_k b\cdot\dot\Delta_k\vec{u}\,dx + 2\|(\dot\Delta_k b,\dot\Delta_k\vec{u})\|_{L^2}^2 + \frac{\varsigma^2}{\gamma}\|(\dot\Delta_k j_0,\dot\Delta_k\vec{j}_1)\|_{L^2}^2.$$

- **Middle frequencies:** Exponential decay for $\|(\dot\Delta_k b,\dot\Delta_k\vec{u},\dot\Delta_k j_0,\dot\Delta_k\vec{j}_1)\|_{L^2}$.

The global existence statement

Theorem (D. & Ducomet, 2013)

Let $\nu := \lambda + 2\mu$. *Under the necessary and sufficient linear stability condition*

$$n^2 \mathcal{C} \nu^2 > B'(1)\,\mathrm{Re}^2\,(2 + \mathcal{L}_s \sigma_s / \sigma_a)$$

there exists a constant c *such that if the data satisfy*

$$\|(\vec{u}^0, j_0^0, \vec{j}_1^0)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|b^0\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|b^0\|^{h}_{\dot{B}^{\frac{n}{2}}_{2,1}} \leq c$$

then our nonlinear system has a unique global solution $(b, \vec{u}, j_0, \vec{j}_1)$ *with*

$$b^\ell \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1}) \quad and \quad b^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1})$$

$$j_0^\ell,\ \vec{j}_1^\ell,\ \vec{u} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1})$$

$$j_0^h,\ \vec{j}_1^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}).$$

Besides, we have the following low frequency decay properties:

$$\Big(j_0 - \frac{\varsigma}{\beta} b - \frac{\varsigma}{\beta^2} \mathrm{div}\,\vec{u} - \frac{\varsigma^2}{\beta^2\gamma} \mathrm{div}\,\vec{j}_1\Big)^\ell \quad and \quad \Big(\vec{j}_1 - \frac{\alpha\varsigma}{\beta\gamma} \nabla b\Big)^\ell \quad in \quad L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}).$$