Compressible Navier-Stokes equations on thin domains

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Abstract and program of the talk

We consider the barotropic Navier-Stokes system describing the motion of a compressible viscous fluid confined to a straight layer $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, where ω is a particular 2-D domain (a periodic cell, bounded domain or the whole $2 - D$ space). We show that the weak solutions in the $3D$ domain converge to a (strong) solutions of the $2 - D$ Navier-Stokes system on ω as $\varepsilon \to 0$ on the maximal life time of the strong solution.

Part 1 : Concept and stability analysis in the continuous case based on the work of E. Feireisl and A. Novotný
Part 2 : Compressible Navier-Stokes equations on thin domains based on the work of D. Maltese, A. Novotný

Compressible barotropic Navier-Stokes equations

We consider in $[0,T) \times \Omega_\epsilon$, $\Omega_\epsilon = \omega \times (0,\epsilon) \subset R^3$ (ω a bounded Lipschitz domain of $\mathbb{R}^2$) the following system of equations

Continuity equation

$$\partial_t \tilde{\varrho}_\epsilon + \mathrm{div}(\tilde{\varrho}_\epsilon \tilde{\mathbf{u}}_\epsilon) = 0 \quad \text{in } (0,T) \times \Omega_\varepsilon,$$

Momentum equation

$$\partial_t(\tilde{\varrho}_\epsilon \tilde{\mathbf{u}}_\epsilon) + \mathrm{div}(\tilde{\varrho}_\epsilon \tilde{\mathbf{u}}_\epsilon \otimes \tilde{\mathbf{u}}_\epsilon) + \nabla p(\tilde{\varrho}_\epsilon) = \mathrm{div}\mathbb{S}(\nabla \tilde{\mathbf{u}}_\epsilon) \quad \text{in } (0,T) \times \Omega_\varepsilon.$$

Boundary conditions

$$\tilde{\mathbf{u}}_\epsilon|_{\partial\omega \times (0,\varepsilon)} = 0,\ \tilde{\mathbf{u}}_\epsilon \cdot \mathbf{n}|_{\omega \times \{0,\varepsilon\}} = 0,\ [\mathbb{S}(\nabla \tilde{\mathbf{u}}_\epsilon) \cdot \mathbf{n}] \times \mathbf{n}|_{\omega \times \{0,\varepsilon\}} = 0.$$

Initial conditions

$$\tilde{\varrho}_\epsilon(0,x) = \tilde{\varrho}_{0,\varepsilon}(x), \quad \tilde{\mathbf{u}}_\epsilon(0,x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x), \quad x \in \Omega_\varepsilon.$$

- $$p \in C[0,\infty) \cap C^2(0,\infty), p(0) = 0, p'(\varrho) > 0$$
- $$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0, \quad \gamma \geq 1$$
- $$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},$$
- $$\varrho H'(\varrho) - H(\varrho) = p(\varrho), \quad H(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s$$

Relative (potential) energy function E

$$E(\varrho|r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$E(\varrho|r) \geq 0, \quad E(\varrho, r) = 0 \Leftrightarrow \varrho = r$$

Rescaling

$$\Omega_\varepsilon \ni (x_h, \varepsilon x_3) \mapsto (x_h, x_3) \in \Omega := \Omega_1, \quad \text{where } x_h = (x_1, x_2),$$

$$\varrho_\varepsilon = \tilde{\varrho}_\varepsilon(x_h, \epsilon x_3), \qquad \mathbf{u}_\varepsilon = \tilde{\mathbf{u}}_\varepsilon(x_h, \varepsilon x_3)$$

Rescaling system :

$$\partial_t \varrho_\varepsilon + \mathrm{div}_\varepsilon(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \quad \text{in } (0,T) \times \Omega,$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_\varepsilon(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_\varepsilon p(\varrho_\varepsilon) = \mathrm{div}_\varepsilon \mathbb{S}(\nabla_\varepsilon \mathbf{u}_\varepsilon) \quad \text{in } (0,T) \times \Omega,$$

$$\varrho_\varepsilon(0,x) = \tilde{\varrho}_{0,\varepsilon}(x_h, \varepsilon x_3) \quad \mathbf{u}(0,x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x_h, \varepsilon x_3), \quad x \in \Omega$$

$$\mathbf{u}_\varepsilon|_{\partial\omega \times (0,1)} = 0, \; \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\omega \times \{0,1\}} = 0, \; [\mathbb{S}(\nabla_\epsilon \mathbf{u}_\varepsilon) \cdot \mathbf{n}] \times \mathbf{n}|_{\omega \times \{0,1\}} = 0.$$

where

$$\nabla_\varepsilon = (\nabla_h, \frac{1}{\varepsilon}\partial_{x_3}), \; \nabla_h = (\partial_{x_1}, \partial_{x_2}),$$

$$\mathrm{div}_\varepsilon \mathbf{u} = \mathrm{div}_h \mathbf{v}_h + \frac{1}{\varepsilon}\partial_{x_3} v_3, \; \mathbf{v}_h = (v_1, v_2), \; \mathrm{div}_h \mathbf{v}_h = \partial_{x_1} v_1 + \partial_{x_2} v_2.$$

Weak solutions

Functional spaces

$$W^{1,2}_{0,\mathbf{n}}(\Omega_\varepsilon;\mathbb{R}^3) \equiv \left\{\mathbf{v}\in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)\,\Big|\,\mathbf{v}|_{\partial\omega\times(0,1)}=0,\ \mathbf{v}\cdot\mathbf{n}|_{\omega\times\{0,1\}}=0\right\}.$$

$\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, $\varrho(t,x)\geq 0$ a.e in $(0,T)\times\Omega$.
$\mathbf{u}\in L^2(0,T;W^{1,2}_{0,\mathbf{n}}(\Omega;\mathbb{R}^3))$.
$\varrho\mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$.

Continuity equation

$\varrho\in C_{\text{weak}}([0,T];L^\gamma(\Omega))$ and the continuity equation is replaced by the family of integral identities

$$\int_{\Omega_\varepsilon}\varrho\varphi\Big|_0^\tau = \int_0^\tau\int_\Omega\Big(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\tau\in[0,T]$ and for any $\varphi\in C^1([0,T]\times\overline{\Omega})$;

Momentum equation

$\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3))$ and momentum equation is satisfied in the sense of distributions, specifically,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \,{\rm d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_\varepsilon\varphi\Big)\,{\rm d}x{\rm d}t$$

$$+ \int_0^\tau \int_\Omega \Big(p(\varrho){\rm div}_\varepsilon\varphi - \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\varphi\Big)\,{\rm d}x\,{\rm d}t$$

for all $\tau \in [0,T]$ and for any
$\varphi \in C_c^\infty([0,T]\times\overline{\Omega};\mathbb{R}^3),\ \varphi|_{[0,T]\times\partial\omega\times(0,1)} = 0,\ \varphi_3|_{[0,T]\times\omega\times\{0,1\}} = 0;$

Energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right]\,{\rm d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\mathbf{u}\,{\rm d}x\,{\rm d}t \le 0$$

holds for a.a. $\tau \in (0,T)$. holds for a.a. $\tau \in (0,T)$. where $\overline{\varrho} > 0$.

Relative energy inequality

Let $(\rho, \mathbf{u})$ be a finite energy weak solution solution of the compressible Navier-Stokes system emmanating from $(\varrho_0, \mathbf{u}_0)$ satisfying.
Energy inequality :

$$\int_\Omega \left(\frac{1}{2}\varrho\mathbf{u}^2 + E(\varrho, \overline{\varrho})\right) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \mathrm{d}t \leq \int_\Omega \left(\frac{1}{2}\varrho_0 \mathbf{u}_0^2 + E(\varrho_0, \overline{\varrho})\right).$$

What happens for

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathrm{U}|^2 + E(\varrho \,|\, r)\right)$$

with $(r, \mathbf{U})$ suitable functions ?

Assume $(\rho, \mathbf{u})$ smooth (and strong) solution.

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho\,|\,r)\right)\Big|_0^\tau + \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}-\mathbf{U})\Big) : \nabla_\varepsilon\Big(\mathbf{u}-\mathbf{U}\Big)\,\mathrm{d}t$$

$$\leq \int_\Omega \varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla_\varepsilon\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{U}) : \nabla_\varepsilon(\mathbf{U}-\mathbf{u})\,\mathrm{d}x$$

$$+ \int_\Omega ((r-\varrho)\partial_t H'(r) + \nabla_\varepsilon H'(r)\cdot(r\mathbf{U}-\varrho\mathbf{u}))\,\mathrm{d}x - \int_\Omega \mathrm{div}_\varepsilon\mathbf{U}\Big(p(\varrho)-p(r)\Big)\,\mathrm{d}x, \quad (1)$$

for (example) all

$$r \in C^1([0,T]\times\overline{\Omega}, \mathbb{R}_+^*),$$

$$\mathbf{U} \in C^1([0,T]\times\overline{\Omega}; R^3),\ \mathbf{U}|_{\partial\omega\times(0,\varepsilon)} = 0,\ U_3|_{\omega\times\{0,\epsilon\}} = 0. \quad (2)$$

And if $(\rho, \mathbf{u})$ is a (finite energy) weak solution ?

Dissipative solutions

Relative entropy

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}) = \int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right)$$

Dissipative (weak) solutions

$(\varrho, \mathbf{u})$ dissipative solution (from $(\varrho_0, \mathbf{u}_0)$) $\Leftrightarrow$ weak solution + above relative energy inequality

Relative energy inequality

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U})(\tau) + \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u} - \mathbf{U})\Big) : \nabla_\varepsilon\Big(\mathbf{u} - \mathbf{U}\Big) \,\mathrm{d}t$$

$$\leq \mathcal{E}(\varrho_0, \mathbf{u}_0 \,\Big|\, r(0), \mathbf{U}(0)) + \int_0^\tau \mathcal{R}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big) \mathrm{d}t$$

where the remainder $\mathcal{R}$ is given by the r.h.s. of formula (1) and the test functions are the same as in formula (2).

Weak solutions versus Dissipative solutions

- Existence of weak solutions ?

- Existence of dissipative solutions ?

- Links between (finite energy) weak solutions and dissipative solutions

-Dissipative solutions $\Rightarrow$ finite energy weak solution

Existence of weak and dissipative solutions

Finite energy initial data

$$\varrho_{0,\varepsilon} \geq 0, \ \int_\Omega \varrho_{0,\varepsilon} = M_\varepsilon > 0, \ \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}^2 + H(\varrho_{0,\varepsilon}) \right) \mathrm{d}x < \infty.$$

Weak solutions : Lions,98 ($\gamma \geq \frac{9}{5}$), Feireisl, Petzeltova, N., 02 ($\gamma > \frac{3}{2}$)

Under assumptions on the initial data and the hypothesis of the pressure with $\gamma > 3/2$, the compressible Navier-Stokes system admits at least one weak solution.

Dissipative solutions : Feireisl, Sun, N., 2011

Under assumptions on initial data and the hypothesis of the pressure with $\gamma > 3/2$, the compressible Navier-Stokes system admits at least one dissipative solution.

Weak solutions are dissipative : Feireisl, Jin, N., 2012

Under assumptions on the initial data and the hypothesis of the pressure with $\gamma > 3/2$, any weak solution of the compressible Navier-Stokes system is a dissipative one.

Red terms at the l.h.s. and at the r.h.s

Energy inequality

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho \mathbf{u}^2 + H(\varrho) \right) \Big|_0^\tau + \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \mathrm{d}t \leq 0 \tag{3}$$

Continuity equation tested by $\frac{1}{2}\mathbf{U}^2$

$$\int_{\Omega_\varepsilon} \frac{1}{2} \varrho \mathbf{U}^2 \Big|_0^\tau = \int_0^\tau \int_{\Omega_\varepsilon} \Big(\varrho \partial_t \mathbf{U} \cdot \mathbf{U} + \varrho \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{U} \Big) \mathrm{dt} \tag{4}$$

Momentum equation tested by $-$

$$- \int_{\Omega_\varepsilon} \varrho \mathbf{u} \cdot \mathbf{U} \Big|_0^\tau = \int_0^\tau \int_{\Omega_\varepsilon} \Big(- \varrho \mathbf{u} \cdot \partial_t \mathbf{U} - \varrho \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{u} + \dots \Big) \mathrm{dt} \tag{5}$$

(3) + (4)+ (5) + algebraic identities gives (1).

Some bibliographic remarks (non exhausting)

Relative entropy method was introduced to the fluid mechanics by Dafermos (90's) in the context of conservation laws. It was broadly used for the Boltzman equation (Golse, Saint-Raymond, Ukai, ...). Inequalities of this type were employed ad-hoc with specific test functions in the case of compressible Navier-Stokes equations for the investigation of low Mach-high Reynolds number limits (Masmoudi, Jiang, Wang, ...). Weak strong uniqueness is a challenging problem mentioned in Lions (98), tempted by Desjardin, 2002 Germain, 2008, who obtained conditional results. The notion of dissipative solutions has been introduced by Lions (98) for the incompressible euler equations, and weak-strong uniqueness in this case has been proved. Theorems on weak solutions, dissipative solutions, relative entropy inequality, weak-strong uniqueness, ... can be obtained also for the complete Navier-Stokes-Fourier system (describing heat conducting flows) in the conservation of energy formulated in terms of the balance of entropy, Feiresl, N., 2009,2012. Can be generalized to unbounded domains and other boundary conditions Jesslé, Jin. Weak-strong uniqueness theorem can be viewed as an "compressible" counterpart of celebrated Prodi-Serrin conditions for incompressible Navier-Stokes equations.

Compressible Navier-Stokes equations on thin domains

- ω an open subset of $\mathbb{R}^2$ and let $\Omega_\epsilon = \omega \times (0, \epsilon)$.

- Ω_ϵ is supposed to be filled with a compressible viscous gas,.

- Whose evolution through the time interval $[0, T]$, $T > 0$ is described by the isentropic compressible Navier-Stokes system for the unknown functions, density $\varrho = \varrho(t, x)$ and velocity $\mathbf{u} = \mathbf{u}(t, x)$, $t \in [0, T]$, $x \in \Omega_\varepsilon$:

$$\partial_t \varrho_\epsilon + \mathrm{div}_\varepsilon(\varrho_\epsilon \mathbf{u}_\epsilon) = 0 \quad \text{in } (0, T) \times \Omega_\varepsilon,$$

$$\partial_t(\varrho_\epsilon \mathbf{u}_\epsilon) + \mathrm{div}_\varepsilon(\varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla_\varepsilon p(\varrho_\epsilon) = \mathrm{div}_\varepsilon \mathbb{S}(\nabla \mathbf{u}_\epsilon) \quad \text{in } (0, T) \times \Omega_\varepsilon.$$

- Equations are completed with the initial conditions

$$\varrho_\epsilon(0, x) = \varrho_{0,\varepsilon}(x), \quad \mathbf{u}_\epsilon(0, x) = \mathbf{u}_{0,\varepsilon}(x), \quad x \in \Omega_\varepsilon$$

- Boundary conditions

$$\mathbf{u}_\epsilon|_{\partial\omega \times (0,1)} = 0, \ \mathbf{u}_\epsilon \cdot \mathbf{n}|_{\omega \times \{0,1\}} = 0, \ [\mathbb{S}(\nabla \ \mathbf{u}_\epsilon) \cdot \mathbf{n}] \times \mathbf{n}|_{\omega \times \{0,1\}} = 0.$$

Goal part 1

- Well prepared initial data :
$[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}](x)$ converge in a certain sense to $[r_0, \mathbf{v}_0](x) = [r_0, \mathbf{v}_{0,h}, 0](x_h)$.

- Investigate the situation when $\varepsilon \to 0$.

Limit expected : the sequence $[\varrho_\varepsilon, \mathbf{u}_\varepsilon](t,x)$ of (weak) solutions will converge to $[r, \mathbf{V}](t, x_h)$, $\mathbf{V} = [\mathbf{w}, 0]$, where the couple $[r(t,x_h), \mathbf{w}(t,x_h)]$ solves :

$$\partial_t r + \mathrm{div}_h(r\mathbf{w}) = 0 \text{ in } (0,T)\times\omega,$$

$$r\partial_t \mathbf{w} + r\mathbf{w}\cdot\nabla_h \mathbf{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h \mathbf{w}) \text{ in } (0,T)\times\omega,$$

$$r(0,x_h) = r_0(x_h),\ \ \mathbf{w}(0,x_h) = \mathbf{w}_0 := \mathbf{v}_{0,h}(x_h),\ \ x_h \in \omega,$$

where

$$\mathbb{S}_h(\nabla_h \mathbf{w}) = \mu\Big(\nabla_h \mathbf{w} + (\nabla_h \mathbf{w})^T - \mathrm{div}_h \mathbf{w}\Big) + (\eta + \frac{\mu}{3})\mathrm{div}_h \mathbf{w}\mathbb{I}_h,$$

and $\mathbb{I}_h$ is the identity matrix.

Goal part 2

-We show that the weak solutions in the $3D$ domain converge to a (strong) solutions of the $2-D$ Navier-Stokes system on ω as $\varepsilon \to 0$ on the maximal life time of the strong solution.

-Tool : Relative energy

- Geometry of ω (+ boundary condition) :
- Periodic layers
- Bounded layers with no-slip boundary
- Bounded layers with slip boundary conditions
- Unbounded layers

Target system

The expected target system is endowed with the no slip boundary conditions :

$$\partial_t r + \mathrm{div}_h(r\mathbf{w}) = 0 \text{ in } (0,T)\times\omega,$$

$$r\partial_t \mathbf{w} + r\mathbf{w}\cdot\nabla_h \mathbf{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h \mathbf{w}) \text{ in } (0,T)\times\omega,$$

$$r(0,x_h) = r_0(x_h),\ \mathbf{w}(0,x_h) = \mathbf{w}_0 := \mathbf{v}_{0,h}(x_h),\ x_h\in\omega,$$

$$\mathbf{w}|_{\partial\omega} = 0.$$

Target system (Valli, Zajaczkowski)

Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$, $\partial\omega \in C^3$. Let

$$r_0 \in W^{2,2}(\omega),\ \inf_{\omega} r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}(\omega;R^2).$$

satisfy the compatibility condition

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big)\Big|_{\partial\omega} = 0.$$

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_{\omega} r_0 \leq D,$$

then the target problem admits a unique strong solution (in the sense a.e. in $(0,T)\times\omega$) in the class

$$r \in C([0,T);W^{2,2}(\omega)),\ \mathbf{w} \in C([0,T);W^{2,2}(\omega;R^2)) \cap L^2(0,T;W^{3,2}(\omega;R^2))$$

$$\partial_t r \in C([0,T);W^{1,2}(\omega)),\ \partial_t \mathbf{w} \in L^2(0,T;W^{2,2}(\omega;R^2)).$$

In particular,

$$0 < \underline{r} \equiv \inf_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \leq \sup_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \equiv \overline{r}.$$

By a density argument we can choose $[r,\mathbf{V}](t,x_h)$, $\mathbf{V} = [\mathbf{w},0]$, where the couple $[r(t,x_h),\mathbf{w}(t,x_h)]$

$$\mathcal{R}(\rho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V})$$

$$= \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\,\mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\,\mathrm{d}x$$

$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\cdot(\mathbf{u}_\varepsilon - \mathbf{V})\,\mathrm{d}x - \int_\Omega (p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r))\mathrm{div}_\varepsilon\mathbf{V}\,\mathrm{d}x.$$

Theorem

Let $\partial\omega \in C^3$. Let r_0, $\mathbf{w}_0$ satisfy assumptions of the previous theorem and let $T_{\max} > 0$ be the life time of the strong solutionof the target problem emmanating from $[r_0, \mathbf{w}_0]$. Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a sequence of weak solutions to the $3-D$ compressible Navier Stokes equations emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that

$$\mathcal{E}\left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \middle| r_0, \mathbf{V}_0\right) \to 0,$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.
Then

$$\text{esssup}_{t\in(0,T_{\max})} \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \big| r, \mathbf{V}) \to 0,$$

*where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $(r, \mathbf{w})$ satsifies the $2-D$ compressible Navier-Stokes system (**??**–18) on the periodic cell ω on the time interval $[0, T_{\max})$.*

Convergence

In order to see more clearly the sense of the limit above, the previous result implies, for example

$$\varrho_\varepsilon \to r \text{ strongly in } L^\infty(0,T;L^\gamma(\Omega)),$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{r}\mathbf{V} \text{ strongly in } L^\infty(0,T;L^2(\Omega;R^3)),$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \to r\mathbf{V} \text{ strongly in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)).$$

Lemma

Let $0 < a < b < \infty$. *Then there exists* $c = c(a,b) > 0$ *such that for all* $\rho \in [0,\infty[$ *and* $r \in [a,b]$ *there holds*

$$E(\rho|r) \geq c(a,b)\Big(1_{\mathbb{R}_+\setminus[\frac{a}{2},2b]} + \rho^\gamma 1_{\mathbb{R}_+\setminus[\frac{a}{2},2b]} + (\rho-r)^2 1_{[\frac{a}{2},2b]}\Big).$$

Application of the relative entropy

We shall estimate the left hand side of the relative entropy inequality with test functions $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ from below by

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{V}\right)(\tau) + c \int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega;R^2)} \mathrm{d}t - c' \int_0^\tau \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \mathbf{V}\right) \mathrm{d}t$$

The right hand side from above by

$$h_\varepsilon(\tau) + \delta \int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega)} \mathrm{d}t + c'(\delta) \int_0^\tau a(t) \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \mathbf{V}\right) \mathrm{d}t$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0, T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0, T).$$

If we succeed...

$$\mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \mathbf{V}\right)(\tau) \le h_\varepsilon(\tau) + c \int_0^\tau a(t) \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \mathbf{V}\right) \mathrm{d}t$$

and we conclude by using the Gronwall's Lemma.

Proof

Lemma

Let $\Omega = \omega \times (0,1)$*, where* ω *is a bounded Lipschitz domain and let* $\eta > 0$*. Then there exists* $c = c(\omega) > 0$ *such that for all* $\mathbf{v} \in W^{1,2}(\Omega; R^3)$*,* $\mathbf{v}|_{\partial\omega \times (0,1)} = 0$ *and* $\varepsilon \in (0,1)$*,*

$$\|\mathbf{v}_h\|^2_{L^2(\Omega,R^2)} + \|\nabla_h \mathbf{v}_h\|^2_{L^2(\Omega,R^{2\times 2})} \leq c \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x,$$

GRAZIE PER LA VOSTRA ATTENZIONE !