Reconstruction scheme for the Euler equations

Nina Aguillon, Christophe Chalons, Frédéric Lagoutière

Université Paris Sud, Université Versailles Saint Quentin

Workshop MODTERCOM "Analyse mathématique et numérique des équations de Navier-Stokes compressibles "

Porquerolles

8-9 septembre 2014

Euler equations

Fluid of density ρ and velocity u.
Isothermal Euler equations: pressure law $p(\rho) = c^2\rho$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + c^2\rho) = 0 \end{cases}$$

Conservation of mass

Newton's law
Conservation of momentum $q := \rho u$

Finite volume scheme ($U = (\rho, q)$)

$$\Delta x\, U_j^{n+1} = \Delta x\, U_j^n + \underbrace{\Delta t\, F_{j-1/2}^n}_{\text{what goes in}} - \underbrace{\Delta t\, F_{j+1/2}^n}_{\text{what goes out}}$$

Numerical viscosity

The Lax-Friedrichs scheme

$$\begin{cases} \rho_j^{n+1} = \frac{\rho_{j-1}^n + \rho_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x}(q_{j+1}^n - q_{j-1}^n) \\ q_j^{n+1} = \frac{q_{j-1}^n + q_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x}\left(\frac{(q_{j+1}^n)^2}{\rho_{j+1}^n} + c^2\rho_{j+1}^n - \frac{(q_{j-1}^n)^2}{\rho_{j-1}^n} - c^2\rho_{j-1}^n\right) \end{cases}$$

approximates at first order the isothermal Euler equations

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2\rho\right) = 0 \end{cases}$$

Numerical viscosity

The Lax-Friedrichs scheme

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{2\Delta x}(q_{j+1}^n - q_{j-1}^n) + \Delta x^2 \frac{\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n}{\Delta x^2} \\ q_j^{n+1} = \frac{q_{j-1}^n + q_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x}\left(\frac{(q_{j+1}^n)^2}{\rho_{j+1}^n} + c^2\rho_{j+1}^n - \frac{(q_{j-1}^n)^2}{\rho_{j-1}^n} - c^2\rho_{j-1}^n \right) \end{cases}$$

approximates at second order the isothermal Euler equations

$$\begin{cases} \partial_t \rho + \partial_x q = \frac{\Delta x^2}{\Delta t}\partial_{xx}\rho - \Delta t \partial_{tt}\rho \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2\rho \right) = \frac{\Delta x^2}{\Delta t}\partial_{xx} q - \Delta t \partial_{tt} q \end{cases}$$

Numerical viscosity

The Lax-Friedrichs scheme

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{2\Delta x}(q_{j+1}^n - q_{j-1}^n) + \Delta x^2 \frac{\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n}{\Delta x^2} \\ q_j^{n+1} = \frac{q_{j-1}^n + q_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x}\left(\frac{(q_{j+1}^n)^2}{\rho_{j+1}^n} + c^2\rho_{j+1}^n - \frac{(q_{j-1}^n)^2}{\rho_{j-1}^n} - c^2\rho_{j-1}^n\right) \end{cases}$$

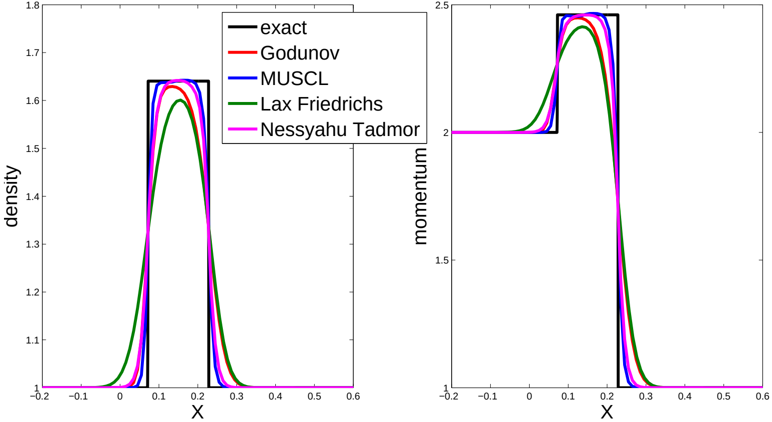
$$\partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \partial_x \begin{pmatrix} q \\ \frac{q^2}{\rho} + c^2\rho \end{pmatrix} = \Delta t \partial_x \left(D_{LF}(\rho, q) \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} \right).$$

$$D_{LF}(\rho, q) = \begin{pmatrix} \frac{\Delta x^2}{\Delta t^2} - (c^2 - u^2) & -2u \\ -2u(c^2 - u^2) & \frac{\Delta x^2}{\Delta t^2} - (3u^2 + c^2) \end{pmatrix}$$

- We recover the CFL condition $(|u| + c)\Delta t \leq \Delta x$.
- The smaller Δt is, the larger the numerical diffusion is.

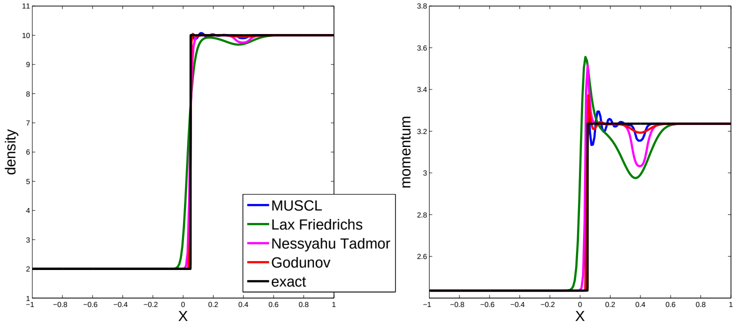
Smoothing effect

The main effect of this numerical viscosity is to smear out the discontinuities of the solution.



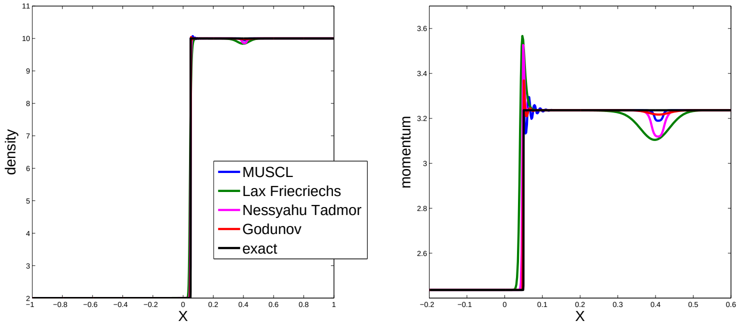
Slowly Moving Shock

This is a shock that travels slowly compared to the speed of sound c (and therefore to $\Delta t/\Delta x$). With 200 cells:



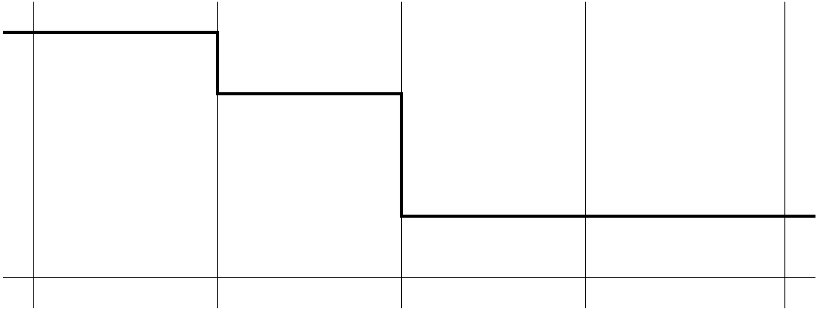
Slowly Moving Shock

This is a shock that travels slowly compared to the speed of sound c (and therefore to $\Delta t/\Delta x$). With 1000 cells:



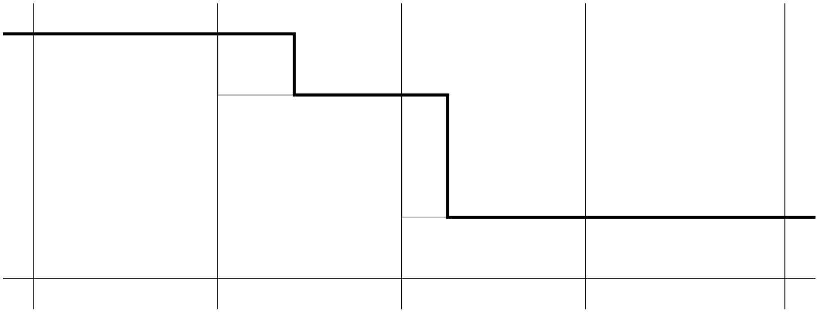
The Godunov scheme on an isolated shock

The numerical solution after the initial sampling



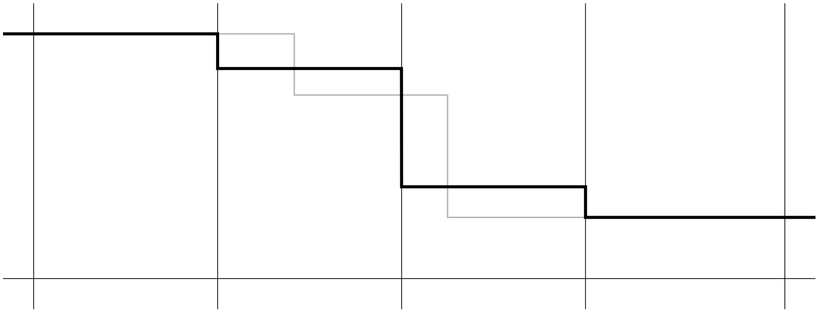
The Godunov scheme on an isolated shock

Compute the exact solution at time Δt



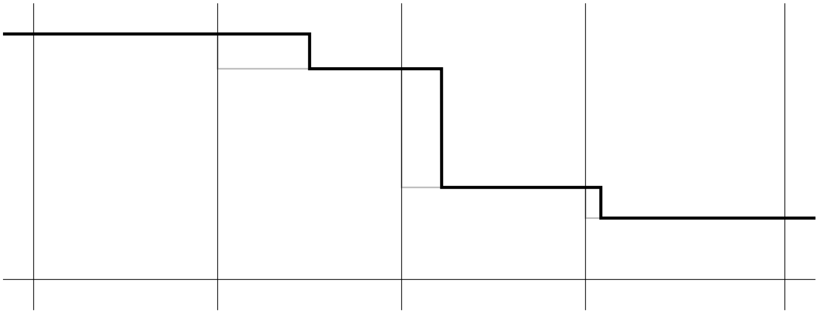
The Godunov scheme on an isolated shock

Average the solution on the mesh

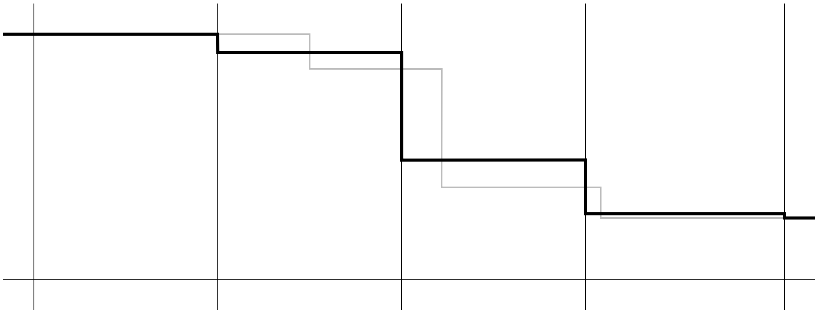


The Godunov scheme on an isolated shock

Compute the exact solution at time Δt



Average the solution on the mesh



The Godunov scheme on an isolated shock

Conclusion

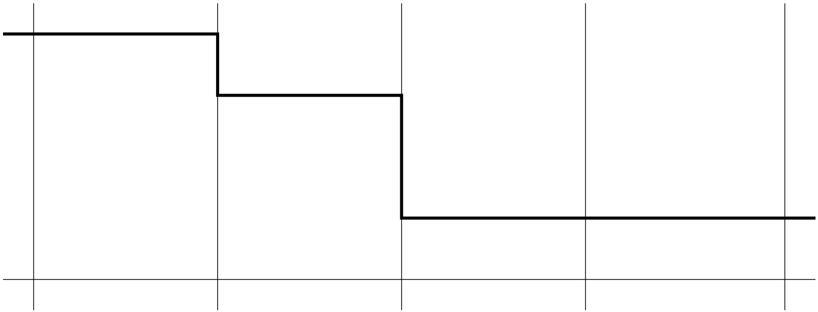
The numerical diffusion comes from the averaging step. Too much information is lost!

Solution

At the beginning of each time step, try to inverse the averaging step.

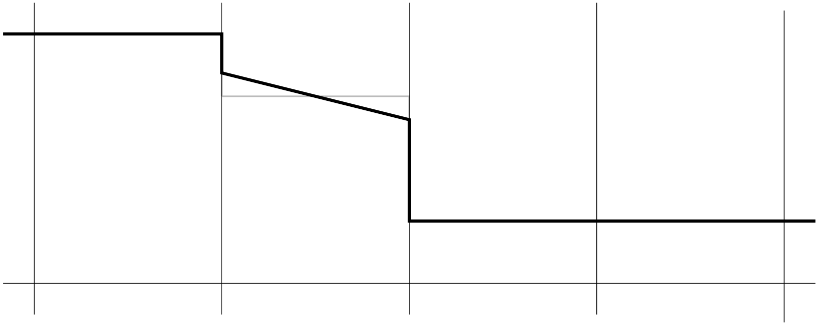
The MUSCL scheme on an isolated shock

The numerical solution after the initial sampling



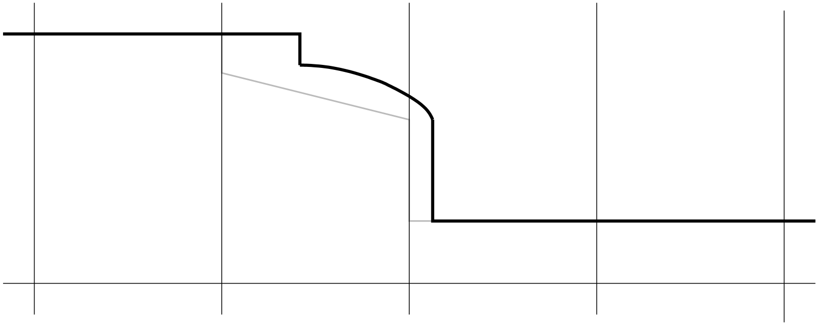
The MUSCL scheme on an isolated shock

Reconstruct the solution as a linear by cell function

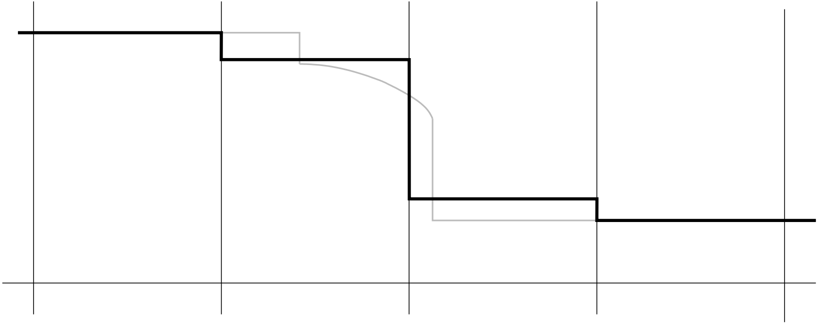


The MUSCL scheme on an isolated shock

Compute the exact solution at time Δt

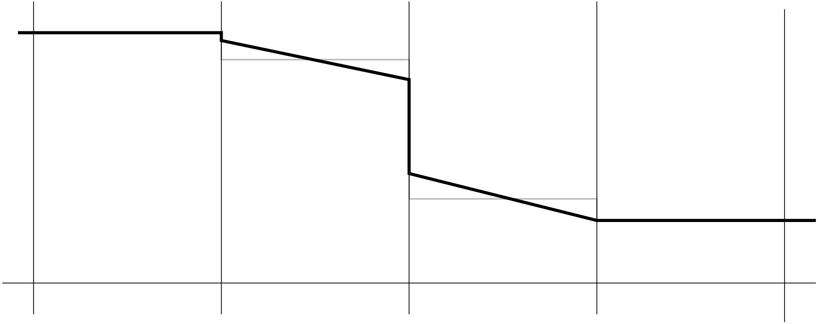


Average the solution on the mesh



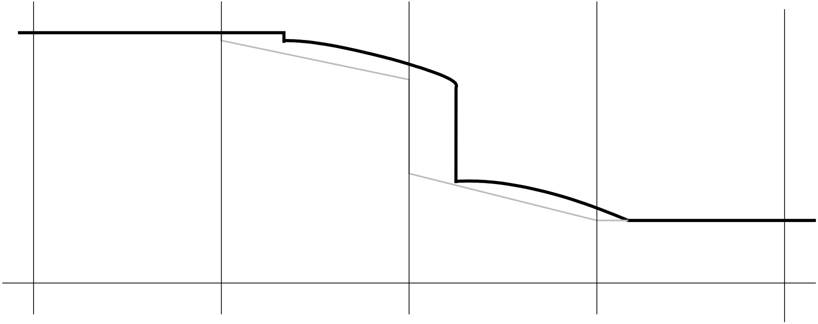
The MUSCL scheme on an isolated shock

Reconstruct the solution as a linear by cell function



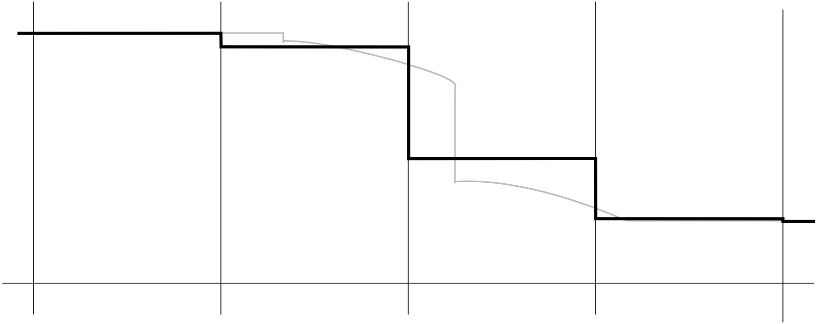
The MUSCL scheme on an isolated shock

Compute the exact solution at time Δt



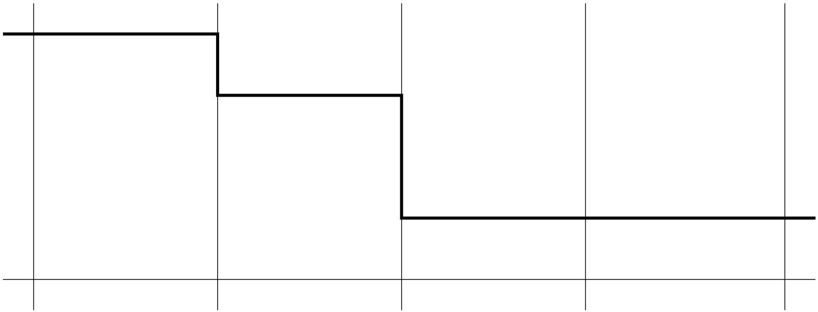
The MUSCL scheme on an isolated shock

Average the solution on the mesh



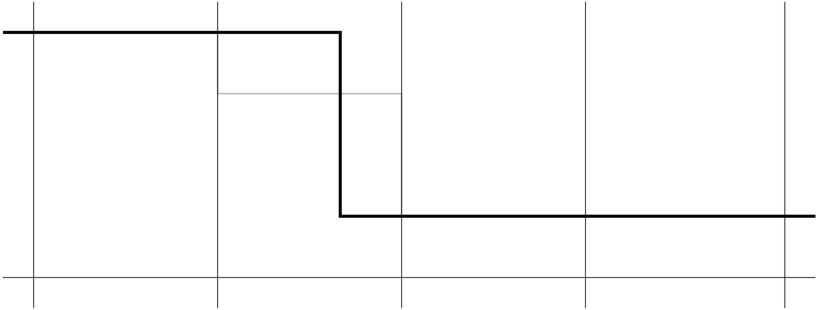
The reconstruction scheme on an isolated shock

The numerical solution after the initial sampling



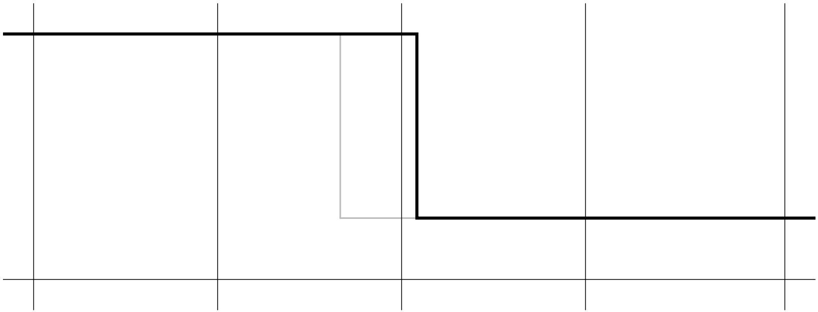
The reconstruction scheme on an isolated shock

Reconstruct the solution as a piecewise constant by cell function



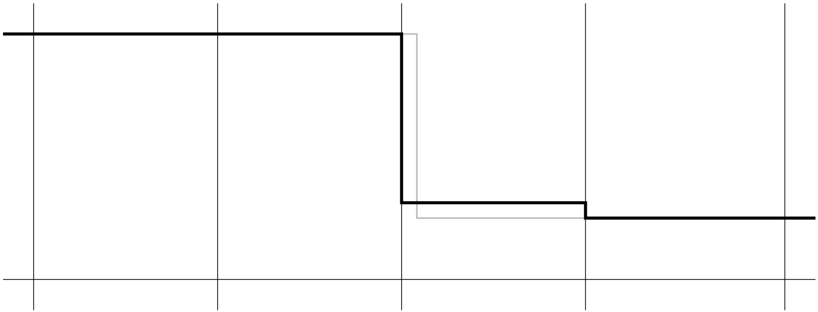
The reconstruction scheme on an isolated shock

Compute the exact solution at time Δt



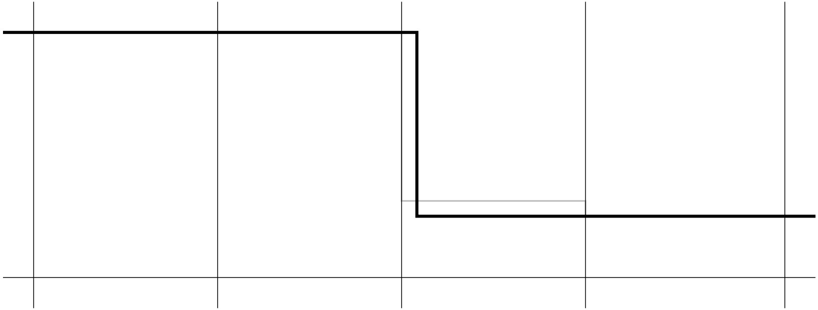
The reconstruction scheme on an isolated shock

Average the solution on the mesh



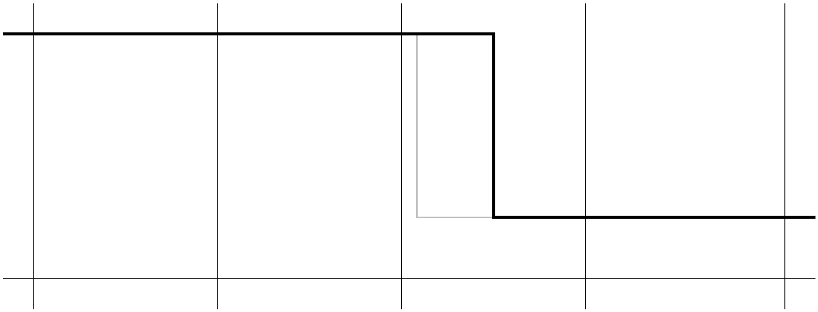
The reconstruction scheme on an isolated shock

Reconstruct the solution as a piecewise constant by cell function



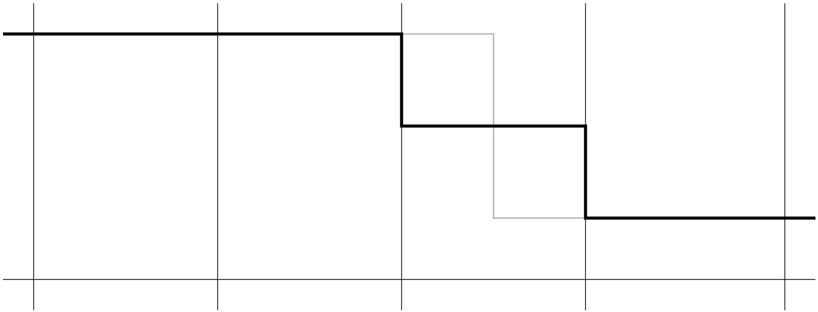
The reconstruction scheme on an isolated shock

Compute the exact solution at time Δt

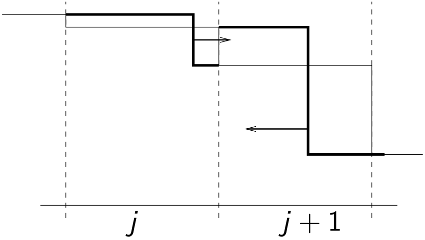


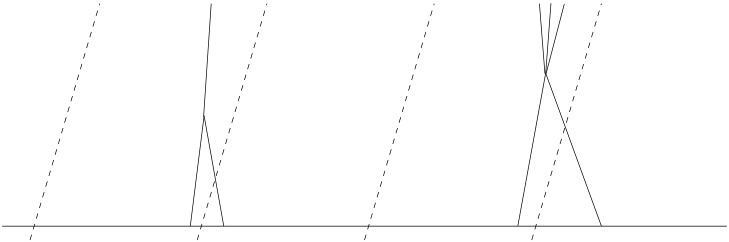
The reconstruction scheme on an isolated shock

Average the solution on the mesh

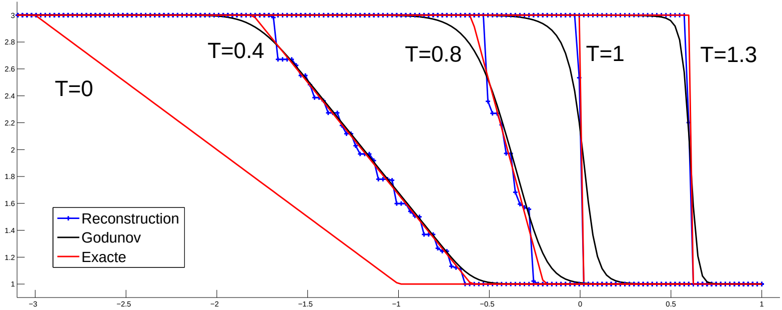


Interaction of shocks and moving mesh



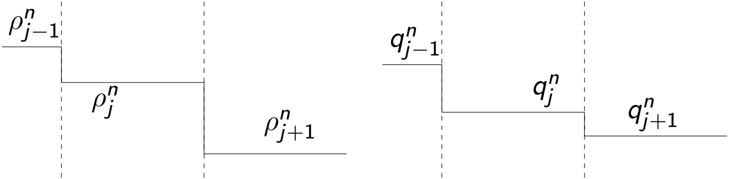


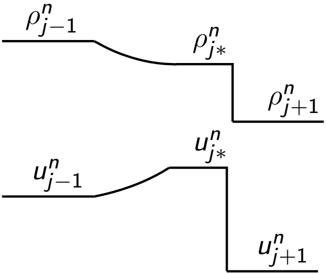
Test case for the Burgers equation



B. Després, F. Lagoutière
Contact Discontinuity Capturing Schemes for Linear Advection and Compressible Gas Dynamics
Journal of Scientific Computing

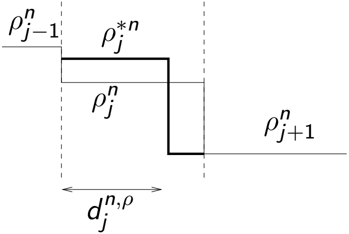
Detection of shocks

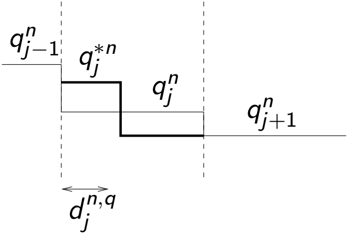


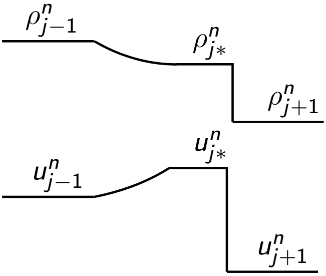


There is no reason for $(\rho^n_{j-1}, q^n_{j-1})$ and $(\rho^n_{j+1}, q^n_{j+1})$ to be linked by a shock!

Detection of shocks



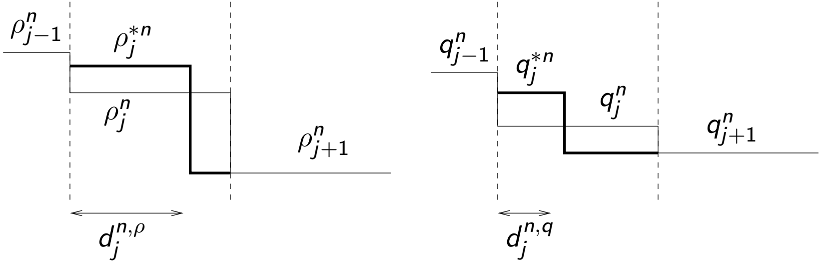




There is no reason for $(\rho^n_{j-1}, q^n_{j-1})$ and $(\rho^n_{j+1}, q^n_{j+1})$ to be linked by a shock!

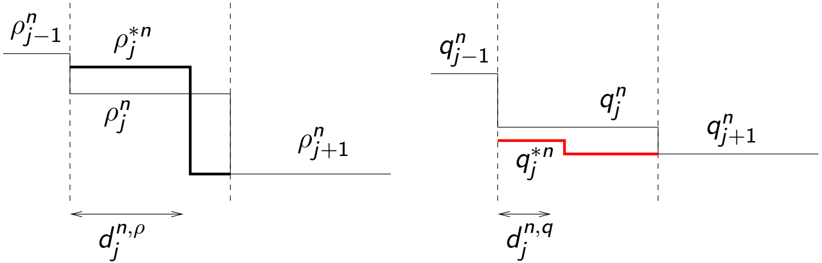
- If $u^n_{j-1} > u^n_{j+1}$ and $\rho^n_{j-1} < \rho^n_{j+1}$ there is a 1-shock in the Riemann problem
- If $u^n_{j-1} > u^n_{j+1}$ and $\rho^n_{j-1} > \rho^n_{j+1}$ there is a 2-shock in the Riemann problem

The scheme



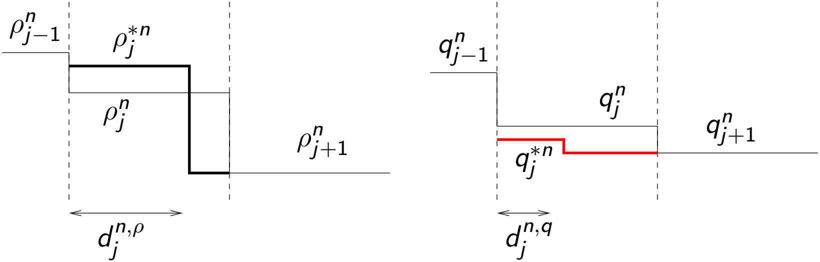
- Detect shocks with the detection lemma, and their associated speed.
- Compute $d_j^{n,\rho}$ and $d_j^{n,q}$ by conservation of ρ and q. Accept the reconstruction if $0 < d_j^{n,\rho} < \Delta x$ and $0 < d_j^{n,q} < \Delta x$.
- Let the reconstructed shock evolve for a time Δt and compute the corresponding flux (easy on a staggered grid)

The scheme



- Detect shocks with the detection lemma, and their associated speed.
- Compute $d_j^{n,\rho}$ and $d_j^{n,q}$ by conservation of ρ and q. Accept the reconstruction if $0 < d_j^{n,\rho} < \Delta x$ and $0 < d_j^{n,q} < \Delta x$.
- Let the reconstructed shock evolve for a time Δt and compute the corresponding flux (easy on a staggered grid)

The scheme



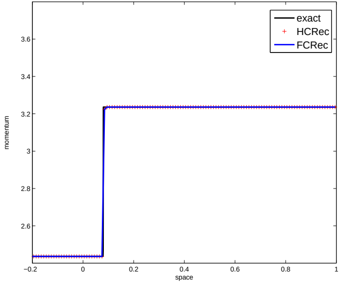
- Detect shocks with the detection lemma, and their associated speed.
- Compute $d_j^{n,\rho}$ and $d_j^{n,q}$ by conservation of ρ and q. Accept the reconstruction if $0 < d_j^{n,\rho} < \Delta x$.
- Let the reconstructed shock evolve for a time Δt and compute the corresponding flux (easy on a staggered grid)

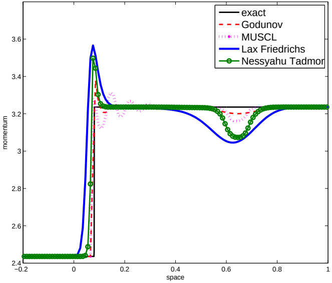
Scheme exact on shocks

The reconstruction scheme is exact when the initial data is a pure shock

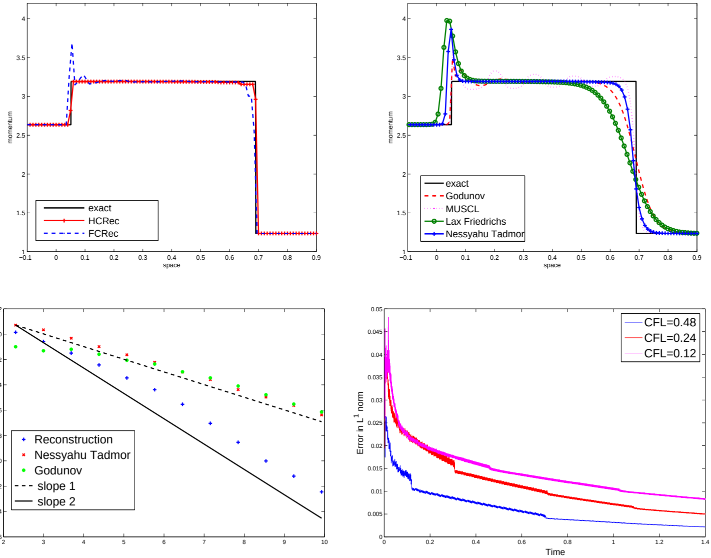
Proof: The reconstruction succeeds only in the crucial cell.

In particular it is exact on slowly moving shocks!





Slowly Moving Shock I



Slowly Moving Shock II

Numerical viscosity

$$\begin{cases} \partial_t \rho + \partial_x q = \varepsilon \partial_{xx} \rho, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2 \rho \right) = \varepsilon \partial_{xx} q \end{cases}$$

S. Jin, J.-G. Liu
The Effects of Numerical Viscosities I. Slowly Moving Shocks
Journal of Computational Physics

Slowly Moving Shock II

Numerical viscosity

Solution on the form $\rho\left(\frac{x-st}{\varepsilon}\right)$, $q\left(\frac{x-st}{\varepsilon}\right)$

$$\begin{cases} \partial_t \rho + \partial_x q = \varepsilon \partial_{xx} \rho, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2 \rho\right) = \varepsilon \partial_{xx} q \end{cases} \qquad \begin{cases} -s\rho' + q' = \rho'' \\ -sq' + \left(\frac{q^2}{\rho} + c^2\rho\right)' = q'' \end{cases}$$

Suppose that ρ increases (1-shock):

$$q = \underbrace{\rho'}_{\text{bump shaped}} + \underbrace{s\rho}_{\text{monotone}} + C$$

Explique l'apparition des chocs lents si la viscosité numérique à cette forme

S. Jin, J.-G. Liu
The Effects of Numerical Viscosities I. Slowly Moving Shocks
Journal of Computational Physics

Slowly Moving Shock II

Physical viscosity (Navier-Stokes)

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2 \rho \right) = \varepsilon \partial_{xx} u \end{cases}$$

S. Jin, J.-G. Liu
The Effects of Numerical Viscosities I. Slowly Moving Shocks
Journal of Computational Physics

Slowly Moving Shock II

Physical viscosity (Navier-Stokes) Solution on the form $\rho\left(\frac{x-st}{\varepsilon}\right)$, $q\left(\frac{x-st}{\varepsilon}\right)$

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2 \rho\right) = \varepsilon \partial_{xx} u \end{cases} \qquad \begin{cases} -s\rho' + q' = 0 \\ -sq' + \left(\frac{q^2}{\rho} + c^2\rho\right)' = \left(\frac{q}{\rho}\right)'' \end{cases}$$

Suppose that ρ increases (1-shock):

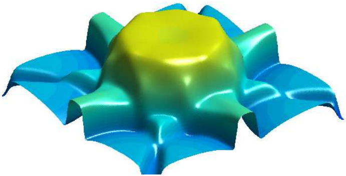
$$q = \underbrace{s\rho}_{\text{monotone}} + C$$

Pas d'apparition des chocs lents si la viscosité numérique à cette forme

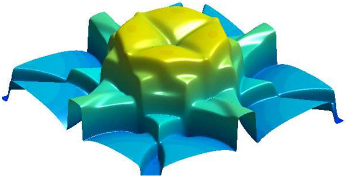
S. Jin, J.-G. Liu
The Effects of Numerical Viscosities I. Slowly Moving Shocks
Journal of Computational Physics

Simulation of a column of water

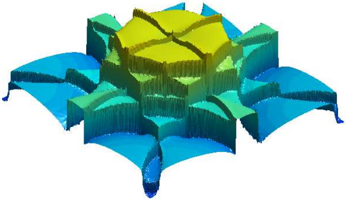
400×400, 3 h



1200×1200, 3 j



400×400, 6 h



Conclusion

The discontinuous reconstruction scheme

- is exact when the initial data is an isolated shock;
- captures sharpely the discontinuities in general;
- behaves very well on some problematic test cases;
- approximates correctly nonclassical solutions.

But...

- contact discontinuities are harder to handle;
- it is possible to use an approximate Riemann solver only if it is exact on isolated shocks;
- the scheme is not entropy satisfying, even though it contains entropy information.

C. Chalons, F. Coquel
Modified Suliciu relaxation system and exact resolution of isolated shock waves
Math. Models Methods Appl. Sci.