Low Mach number limit in a L^p type framework

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Barotropic Navier-Stokes equations:

$$(NSC) \qquad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) - \mu\Delta u - \mu'\nabla \mathrm{div}\, u + \nabla P = 0 \end{cases}$$

with

- $\rho = \rho(t,x) \in \mathbb{R}^+$: density,
- $u = u(t,x) \in \mathbb{R}^d$: velocity field,
- $P = P(\rho)$: given pressure function,
- $\mu > 0$ and $\mu + \mu' > 0$, given viscosity coefficients,
- Fluid domain may be $\mathbb{R}^d$, $\mathbb{T}^d$, $\mathbb{T}\times\mathbb{R}^{d-1}$, etc.

Models and formal analysis

Rescaled barotropic Navier-Stokes equations with Mach number ε :

$$(NSC_\varepsilon) \quad \begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu\Delta u^\varepsilon - (\lambda+\mu)\nabla \operatorname{div} u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} = 0. \end{cases}$$

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Formal limit system : **incompressible Navier-Stokes equations**:

$$(NS) \quad \begin{cases} \rho = cste, \\ \mathrm{div}\, v = 0, \\ \partial_t v + v \cdot \nabla v - \mu\Delta v + \nabla\Pi = 0. \end{cases}$$

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Ill-prepared data: $\rho_0^\varepsilon = 1 + \varepsilon b_0^\varepsilon$ and u_0^ε with (b_0^ε) and (u_0^ε) suitably bounded.

Main issue: Can we justify existence and convergence in a L^p type framework ?
For simplicity we consider only the small data / global solutions case.

The limit system

Introducing $\mathcal{P} := \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$, System (NS) recasts in

$$\partial_t u + \mathcal{P}\mathrm{div}\,(u \otimes u) - \mu\Delta u = 0$$

which is formally equivalent to $u = v_0 + \mathcal{B}(u,u)$ with

$$v_0(t) := e^{\mu t\Delta}u_0 \quad \text{and} \quad \mathcal{B}(u,v)(t) := -\int_0^t e^{\mu(t-\tau)\Delta}\mathcal{P}\mathrm{div}\,(u\otimes v)\,d\tau.$$

This is a fixed point problem that may be solved in a number of *scaling invariant spaces for (NS)*, that is v is a solution to (NS) if and only if $T_\lambda v$ is a solution (for all $\lambda > 0$) with

$$T_\lambda v(t,x) := \lambda v(\lambda^2 t, \lambda x).$$

Theorem (Fujita-Kato, Kozono-Yamazaki, Cannone-Meyer-Planchon, Chemin . . .)

Let $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,r}$ *with* $\mathrm{div}\, u_0 = 0$ *and* $p < \infty$. *There exists* $c > 0$ *such that if*

$$\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} \le c\mu$$

then (NS) *has a unique global solution* u *in the space*

$$X := \widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \widetilde{L}^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+1}_{p,r}).$$

Setting $\rho^\varepsilon = 1 + \varepsilon a^\varepsilon$, the barotropic Navier-Stokes system rewrites

$$(NSC_\varepsilon) \qquad \begin{cases} \partial_t a^\varepsilon + u^\varepsilon \cdot \nabla a^\varepsilon = -\varepsilon^{-1}(1+\varepsilon a^\varepsilon)\mathrm{div}\, u^\varepsilon, \\ \partial_t u^\varepsilon - \mathcal{A}u^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon - J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon - \varepsilon^{-1}\nabla G(\varepsilon a^\varepsilon) \end{cases}$$

with $\mathcal{A} := \mu\Delta - \mu'\nabla\mathrm{div}$, $J(b) := b/(1+b)$ and $G'(b) = P'(b)/(1+b)$.

- Local-in-time existence results of C^∞ solutions : J. Serrin (1959), J. Nash (1962).
- Local strong solutions in parabolic Sobolev spaces: V. Solonnikov (1976), A. Valli and W. Zajączkowski (1986). Global if small data and strictly convex increasing pressure law: P.B. Mucha and W. Zajączkowski (2002).

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- Local well-posedness in critical Besov spaces : $a_0^\varepsilon \in \dot{B}^{d/p}_{p,1}$ and $u_0^\varepsilon \in \dot{B}^{d/p-1}_{p,1}$ with $1 \leq p < 2d$ (R.D. (2001, 12)).

  NB: In this context, critical means invariance by

  $$a^\varepsilon(t,x) \to a^\varepsilon(\lambda^2 t, \lambda x), \qquad u^\varepsilon(t,x) \to \lambda u^\varepsilon(\lambda^2 t, \lambda x),$$

  which leaves System (NSC_ε) invariant if the pressure is neglected.

The proof is a combination of standard estimates for the heat and transport equations. Totally independent of energy arguments.

Aim: *global* existence theory in a L^p framework consistent with the limit system. Typically, we would like $\mathcal{P}u_0^\varepsilon \in \dot{B}^{d/p-1}_{p,r}$.

The linearized equations about $(a,u)=(0,0)$

Stability assumption : $P'(1)=1$.

Helmholtz decomposition : $u^\varepsilon = \mathcal{P}u^\varepsilon + \mathcal{Q}u^\varepsilon$. Total viscosity: $\nu := \mu + \mu'$.

Linearized equations :
$$\begin{cases} \partial_t b^\varepsilon + \dfrac{\operatorname{div}\mathcal{Q}u^\varepsilon}{\varepsilon} = f^\varepsilon, \\ \partial_t \mathcal{Q}u^\varepsilon - \nu\Delta\mathcal{Q}u^\varepsilon + \dfrac{\nabla b^\varepsilon}{\varepsilon} = g^\varepsilon, \\ \partial_t \mathcal{P}u^\varepsilon - \mu\Delta\mathcal{P}u^\varepsilon = h^\varepsilon. \end{cases}$$

At the linear level, a L^p approach is thus relevant for $\mathcal{P}u^\varepsilon$.

Denoting $v^\varepsilon := |D|^{-1}\operatorname{div}\mathcal{Q}u^\varepsilon$, the first two equations are equivalent to

$$(BM_\varepsilon) \qquad \begin{cases} \partial_t b^\varepsilon + \dfrac{|D|v^\varepsilon}{\varepsilon} = f^\varepsilon, \\ \partial_t v^\varepsilon - \nu\Delta v^\varepsilon - \dfrac{|D|v^\varepsilon}{\varepsilon} = \widetilde{g}^\varepsilon. \end{cases}$$

Spectral analysis of (BM_ε)

- Low frequency regime $\nu\varepsilon|\xi| < 2$: Eigenvalues read

$$\lambda^{\pm} = -\frac{\nu|\xi|^2}{2}\left(1 \pm i\sqrt{\frac{4}{\varepsilon^2\nu^2|\xi|^2} - 1}\right).$$

As $\lambda \pm (\xi) \sim -\nu\frac{|\xi|^2}{2} \mp i\frac{|\xi|}{\varepsilon}$ for $\xi \to 0$, we expect (BM_ε) to behave like

$$\frac{d}{dt}z - \frac{\nu}{2}\Delta z \mp i\frac{|D|}{\varepsilon}z = 0.$$

Because of imaginary part, energy type framework is mandatory.

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$$\frac{d}{dt}z - \frac{\nu}{2}\Delta z \mp i\frac{|D|}{\varepsilon}z = 0.$$

Because of imaginary part, energy type framework is mandatory.

- High frequency regime $\nu\varepsilon|\xi| > 2$: Eigenvalues read

$$\lambda^\pm = -\frac{\nu|\xi|^2}{2}\left(1 \pm \sqrt{1 - \frac{4}{\varepsilon^2\nu^2|\xi|^2}}\right).$$

For $|\xi| \to \infty$, we get

  - $\lambda^+ \sim -\nu|\xi|^2$: parabolic mode with diffusion ν;
  - $\lambda^- \sim -\frac{1}{\varepsilon^2\nu}$: damped mode.

There is no obstruction for L^p framework in the high frequency regime.

Global well-posedness in the case $\varepsilon=\nu=1$

Theorem (Charve-D., Chen-Miao-Zhang, Haspot)

Let $p\in[2,2d[\cap[2,\min(4,\frac{2d}{d-2})]$. *Assume* $P'(1)>0$, $a_0\in\dot B^{\frac dp}_{p,1}$ *and* $u_0\in\dot B^{\frac dp-1}_{p,1}$ *and that* a_0^ℓ *and* u_0^ℓ *are in* $\dot B^{\frac d2-1}_{2,1}$. *There exist two constants* c *and* M *s. t. if*

$$\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}}\leq c$$

then (NSC) *has a unique global-in-time solution* (a,u) *with*

$$(a,u)^\ell\in\mathcal C_b(\mathbb R^+;\dot B^{\frac d2-1}_{2,1})\cap L^1(\mathbb R^+;\dot B^{\frac d2+1}_{2,1}),\quad a^h\in\mathcal C_b(\mathbb R^+;\dot B^{\frac dp}_{p,1})\cap L^1(\mathbb R^+;\dot B^{\frac dp}_{p,1}),$$
$$u^h\in\mathcal C_b(\mathbb R^+;\dot B^{\frac dp-1}_{p,1})\cap L^1(\mathbb R^+;\dot B^{\frac dp+1}_{p,1}).$$

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Theorem (Charve-D., Chen-Miao-Zhang, Haspot)

Let $p \in [2, 2d[\cap[2, \min(4, \frac{2d}{d-2})]$. Assume $P'(1) > 0$, $a_0 \in \dot{B}^{\frac{d}{p}}_{p,1}$ and $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$ and that a_0^ℓ and u_0^ℓ are in $\dot{B}^{\frac{d}{2}-1}_{2,1}$. There exist two constants c and M s. t. if

$$\|(a_0, u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq c$$

then (NSC) has a unique global-in-time solution (a, u) with

$$(a, u)^\ell \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \quad a^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}}_{p,1}),$$
$$u^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+1}_{p,1}).$$

We want to get the following improvements:

1. Weaker condition on $\mathcal{P}u_0$ ($\mathcal{P}u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,r}$ for some $r > 1$),
2. Get the result for all small ε with 'uniform' bounds,
3. Convergence to the incompressible Navier-Stokes equations.

General strategy

Step 1. Global existence with uniform estimates in a critical L^p type framework

The right functional framework may be guessed from linear analysis, (endpoint) maximal regularity and scaling considerations :

- Low frequencies $\varepsilon\nu|\xi| < 1$: assume $(a_0^\varepsilon, \mathcal{Q}u_0^\varepsilon)^\ell \in \dot{B}_{2,1}^{\frac{d}{2}-1}$.
- High frequencies $\varepsilon\nu|\xi| > 1$: assume $(a_0^\varepsilon, \mathcal{Q}u_0^\varepsilon)^h \in \dot{B}_{p,1}^{\frac{d}{p}} \times \dot{B}_{p,1}^{\frac{d}{p}-1}$.
- $\mathcal{P}u_0^\varepsilon$ only in the larger space $\dot{B}_{p,r}^{\frac{d}{p}-1}$.

Existence space $X_{\varepsilon,\nu}^p$ given by the following norm :

$$\|(a,u)\|_{X^p_{\varepsilon,\nu}} := \|(a,\mathcal{Q}u)^\ell\|_{\widetilde{L}^\infty(\dot{B}^{d/2-1}_{2,1})} + \|\mathcal{Q}u^h\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})} + \|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r}\cap \dot{B}^{-1}_{\infty,1})}$$
$$+\varepsilon\nu\|a^h\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})} + \nu\|(a,\mathcal{Q}u)^\ell\|_{L^1(\dot{B}^{d/2+1}_{2,1})}$$
$$+\nu\|\mathcal{Q}u^h\|_{L^1(\dot{B}^{d/p+1}_{p,1})} + \mu\|\mathcal{P}u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r}\cap \dot{B}^{1}_{\infty,1})} + \varepsilon^{-1}\|a^h\|_{L^1(\dot{B}^{d/p}_{p,1})}.$$

Reduction to $\nu = 1$ **and** $\varepsilon = 1$ thanks to the following rescaling :

$$(a,u)(t,x) := \varepsilon(a^\varepsilon, u^\varepsilon)(\varepsilon^2\nu t, \varepsilon\nu x) \quad \text{and} \quad (a_0,u_0)(x) := \varepsilon(a_0^\varepsilon, u_0^\varepsilon)(\varepsilon\nu x)$$

Step 2. Proof of convergence.

Global a priori estimates for $\varepsilon = \nu = 1$ (based on Haspot's method)

1. Parabolic estimates for $\mathcal{P}u$. Apply endpoint maximal regularity estimates to

$$\partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = -\mathcal{P}(J(a)\mathcal{A}u) - \mathcal{P}(u\cdot\nabla u).$$

Product laws yield if $1 \leq p < 2d$:

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{p}-1}_{p,r})} + \mu\|\mathcal{P}u\|_{L^1(\dot{B}^{\frac{d}{p}+1}_{p,r})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} + \|(a,u)\|^2_{X^p_{1,1}},$$

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})} + \mu\|\mathcal{P}u\|_{L^1(\dot{B}^{1}_{\infty,1})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{-1}_{\infty,1}} + \|(a,u)\|^2_{X^p_{1,1}}.$$

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Product laws yield if $1\leq p<2d$:
$$\|\mathcal{P}u\|_{\widetilde L^\infty(\dot B^{\frac dp-1}_{p,r})}+\mu\|\mathcal{P}u\|_{L^1(\dot B^{\frac dp+1}_{p,r})}\lesssim\|\mathcal{P}u_0\|_{\dot B^{\frac dp-1}_{p,r}}+\|(a,u)\|^2_{X^p_{1,1}},$$
$$\|\mathcal{P}u\|_{\widetilde L^\infty(\dot B^{-1}_{\infty,1})}+\mu\|\mathcal{P}u\|_{L^1(\dot B^{1}_{\infty,1})}\lesssim\|\mathcal{P}u_0\|_{\dot B^{-1}_{\infty,1}}+\|(a,u)\|^2_{X^p_{1,1}}.$$

2. Low frequency estimates for $(a,\mathcal{Q}u)$. Remember that
$$\begin{cases}\partial_t a+\operatorname{div}\mathcal{Q}u=-\operatorname{div}(au),\\ \partial_t\mathcal{Q}u-\Delta\mathcal{Q}u+\nabla a=-\mathcal{Q}(u\cdot\nabla u)-\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)+k(a)\nabla a.\end{cases}\qquad(1)$$
(Spectrally localized) energy estimates yield
$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde L^\infty(\dot B^{d/2-1}_{2,1})\cap L^1(\dot B^{d/2+1}_{2,1})}\lesssim\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot B^{d/2-1}_{2,1}}+\|\operatorname{div}(au)\|^\ell_{L^1(\dot B^{d/2-1}_{2,1})}$$
$$+\|\mathcal{Q}(u\cdot\nabla u)\|^\ell_{L^1(\dot B^{d/2-1}_{2,1})}+\|\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)\|^\ell_{L^1(\dot B^{d/2-1}_{2,1})}+\|k(a)\nabla a\|^\ell_{L^1(\dot B^{d/2-1}_{2,1})},$$
and thus
$$\|(a,u)\|^\ell_{\widetilde L^\infty(\dot B^{\frac d2-1}_{2,1})}+\|(a,u)\|^\ell_{L^1(\dot B^{\frac d2+1}_{2,1})}\lesssim\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|(a,u)\|^2_{X^p_{1,1}}.$$

3. Effective velocity $w := \mathcal{Q}u + (-\Delta)^{-1}\nabla a$. (That is $\Delta w = \Delta\mathcal{Q}u - \nabla a$).

$$\partial_t w - \Delta w = -\mathcal{Q}(u\cdot\nabla u) - \mathcal{Q}(J(a)\widetilde{\mathcal{A}}u) + k(a)\nabla a + \mathcal{Q}(au) + w - (-\Delta)^{-1}\nabla a.$$

Applying the heat estimates for the high frequencies of w yields

$$\|w\|^h_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})\cap L^1(\dot{B}^{d/p+1}_{p,1})} \lesssim \|w_0\|^h_{\dot{B}^{d/p-1}_{p,1}} + \|u\cdot\nabla u\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})} + \|J(a)\widetilde{\mathcal{A}}u\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}$$
$$+\|k(a)\nabla a\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})} + \|\mathcal{Q}(au)\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})} + \|w\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})} + \|a\|^h_{L^1(\dot{B}^{d/p-2}_{p,1})}.$$

The important point is that if the high frequency cut-off is at $|\xi| = H$,

$$\|z\|^h_{L^1(\dot{B}^{d/p-\alpha}_{p,1})} \lesssim H^{-2}\|z\|^h_{L^1(\dot{B}^{d/p-\alpha+2}_{p,1})}.$$

At the end we get

$$\|w\|^h_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})\cap L^1(\dot{B}^{d/p+1}_{p,1})} \lesssim \|w_0\|^h_{\dot{B}^{d/p-1}_{p,1}} + H\|(a,u)\|^2_{X^p_{1,1}} + H^{-2}\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})}.$$

Global a priori estimates in the case $\varepsilon=\nu=1$ (end)

4. High frequencies of the density : satisfy a transport equation with damping :

$$\partial_t a + u\cdot\nabla a + a = -a\,\mathrm{div}\,u - \mathrm{div}\,w.$$

We find out that

$$\|a(t)\|^h_{\dot B^{d/p}_{p,1}} + \int_0^t \|a\|^h_{\dot B^{d/p}_{p,1}}\,d\tau \lesssim \|a_0\|^h_{\dot B^{d/p}_{p,1}} + \int_0^t \bigl(\|\nabla u\|_{L^\infty} + \|\mathrm{div}\,u\|_{\dot B^{d/p}_{p,1}}\bigr)\|a\|_{\dot B^{d/p}_{p,1}}\,d\tau$$
$$+\|\nabla a\|_{\widetilde L^\infty(\dot B^{d/p-1}_{p,1})}\|u\|_{\widetilde L^1(\dot B^{d/p+1}_{p,r})} + \|w\|^h_{L^1(\dot B^{d/p+1}_{p,1})}.$$

Red term requires a control on ∇u in $L^1(\mathbb{R}_+;L^\infty)$. This explains why we have to assume in addition that $\mathcal{P}u_0\in\dot B^{-1}_{\infty,1}$.

At the end, we get

$$\|a\|^h_{(L^1\cap\widetilde L^\infty)(\dot B^{d/p}_{p,1})} \lesssim \|a_0\|^h_{\dot B^{d/p}_{p,1}} + \|\mathcal{Q}u_0\|^h_{\dot B^{d/p-1}_{p,1}} + H\|(a,u)\|^2_{X^p_{1,1}}.$$

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5. Conclusion. Take H so large as to absorb $H^{-2}\|a\|^h_{L^1(\dot B^{d/p}_{p,1})}$. We get

$$\|(a,u)\|_{X^p_{1,1}} \le C\Big(\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot B^{d/2-1}_{2,1}} + \|\mathcal{P}u_0\|_{\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}} + \|a_0\|^h_{\dot B^{d/p}_{p,1}}$$
$$+\|\mathcal{Q}u_0\|^h_{\dot B^{d/p-1}_{p,1}} + \|(a,u)\|^2_{X^p_{1,1}}\Big).$$

It is then easy to close the estimates globally if the data are small.

Proof of convergence by compactness

Up to subsequence, $(a^\varepsilon, u^\varepsilon) \rightharpoonup (a,u)$ in $\widetilde{L}^\infty(\mathbb{R}_+; \dot{B}^{\frac{d}{p}-1}_{p,r})$ weakly $*$.

- As $\rho^\varepsilon = 1 + \varepsilon a^\varepsilon$, it is obvious that $\rho^\varepsilon \to 1$.
- Likewise, $\operatorname{div} u^\varepsilon = -\varepsilon \operatorname{div}(a^\varepsilon u^\varepsilon) - \varepsilon \partial_t a^\varepsilon$ implies $\operatorname{div} u = 0$.
- $\partial_t \mathcal{P} u^{\varepsilon_n} - \mu \Delta \mathcal{P} u^{\varepsilon_n} = -\mathcal{P}(u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n}) - \mathcal{P}\big(J(\varepsilon_n a^{\varepsilon_n}) \mathcal{A} u^{\varepsilon_n}\big)$ and

$$u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n} = \frac{1}{2} \nabla |\mathcal{Q} u^{\varepsilon_n}|^2 + \mathcal{P} u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n} + \mathcal{Q} u^{\varepsilon_n} \cdot \nabla \mathcal{P} u^{\varepsilon_n}.$$

Hence it suffices to prove that

$$\mathcal{P}(\mathcal{P} u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n}) \rightharpoonup \mathcal{P}(\mathcal{P} u \cdot \nabla u) \quad \text{and} \quad \mathcal{P}(\mathcal{Q} u^{\varepsilon_n} \cdot \nabla \mathcal{P} u^{\varepsilon_n}) \rightharpoonup 0. \qquad (2)$$

We already know that $\mathcal{Q} u = 0$.

Uniform bounds on time derivatives, and compact embedding yields

$$\phi \mathcal{P} u^{\varepsilon_n} \longrightarrow \phi \mathcal{P} u \quad \text{in} \quad \mathcal{C}([0,T]; \dot{B}^{\frac{d}{p}-\frac{3}{2}}_{p,r}) \quad \text{for all} \quad \phi \in \mathcal{C}^\infty_0(\mathbb{R}^d),$$

which, combined with uniform bounds, ensures (2).

Other proofs of convergence : Strichartz estimates ($\mathbb{R}^d$ case) or Schochet filtering method ($\mathbb{T}^d$ case).

Theorem (R.D. and L. He, 2014)

Assume that the fluid domain is either $\mathbb{R}^d$ *or* $\mathbb{T}^d$ $(d=2,3)$, *that* $2\leq p\leq 4$ *and* $1\leq r\leq p/(p-2)$ *(with* $p\neq 4$ *if* $d=2$*). There exists* η *s.t. for all* $\varepsilon,\nu>0$, *if*

$$\|(a_0^\varepsilon,\mathcal{Q}u_0^\varepsilon)\|^\ell_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{Q}u_0^\varepsilon\|^h_{\dot B^{d/p-1}_{p,1}}+\|\mathcal{P}u_0^\varepsilon\|_{\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}}+\varepsilon\nu\|a_0^\varepsilon\|^h_{\dot B^{d/p}_{p,1}}\leq\eta\nu,$$

then (NSC_ε) *with initial data* $(a_0^\varepsilon,u_0^\varepsilon)$ *has a unique global solution* $(a^\varepsilon,u^\varepsilon)$ *in the space* $X^p_{\varepsilon,\nu}$ *with, for some constant* C *independent of* ε *and* ν,

$$\|(a^\varepsilon,u^\varepsilon)\|_{X^p_{\varepsilon,\nu}}\leq C\Big(\|(a_0^\varepsilon,\mathcal{Q}u_0^\varepsilon)\|^\ell_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{Q}u_0^\varepsilon\|^h_{\dot B^{d/p-1}_{p,1}}+\|\mathcal{P}u_0^\varepsilon\|_{\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}}+\varepsilon\nu\|a_0^\varepsilon\|^h_{\dot B^{d/p}_{p,1}}\Big).$$

In addition, $\mathcal{Q}u^\varepsilon\rightharpoonup 0$ *and, if* $\mathcal{P}u_0^\varepsilon\rightharpoonup v_0$ *then* $\mathcal{P}u^\varepsilon$ *tends weakly to the corresponding solution* v *to* (NS).

We have $(a^\varepsilon,\mathcal{Q}u^\varepsilon)\to 0$ *and* $\mathcal{P}u^\varepsilon\to v$ *strongly (for suitable norms) if the fluid domain is* $\mathbb{R}^d$ *and if* $\mathcal{P}u_0^\varepsilon\to v_0$.