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Low Mach number limit and diffusion limits in a model of radiative flow

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A mathematical model of radiative flow

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u}\otimes \vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S} + \vec{S}_F \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t \left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\right) + \mathrm{div}_x\left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\vec{u}\right) \tag{1.3}$$

$$+\mathrm{div}_x\left(p\vec{u} + \vec{q} - \mathbb{S}\vec{u}\right) = -S_E \quad \text{in } (0,T)\times\Omega,$$

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.4}$$

the mass density $\varrho = \varrho(t, x)$,

the velocity field $\vec{u} = \vec{u}(t, x)$,

the temperature $\vartheta = \vartheta(t, x)$

the radiative intensity $I = I(t, x, \vec{\omega}, \nu)$, depending on the direction $\vec{\omega} \in \mathcal{S}^2$,

$\mathcal{S}^2 \subset \mathbb{R}^3$ the unit sphere,

the frequency $\nu \geq 0$.

$p = p(\varrho, \vartheta)$ the thermodynamic pressure

$e = e(\varrho, \vartheta)$ is the specific internal energy

Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho, \vartheta) - \vartheta \frac{\partial p}{\partial \vartheta} \right). \qquad (1.5)$$

$\mathbb{S}$ is the viscous stress tensor

$$\mathbb{S} = \mu \left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3} \mathrm{div}_x \vec{u} \right) + \eta \ \mathrm{div}_x \vec{u} \ \mathbb{I}, \qquad (1.6)$$

the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$

the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature

$\vec{q}$ is the heat flux given by Fourier's law

$$\vec{q} = -\kappa \nabla_x \vartheta, \tag{1.7}$$

the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

$$S = S_{a,e} + S_s, \tag{1.8}$$

$$S_{a,e} = \sigma_a \Big(B(\nu, \vartheta) - I \Big), \ S_s = \sigma_s \left(\tilde{I} - I \right). \tag{1.9}$$

$$S_E = \int_{\mathcal{S}^2} \int_0^\infty S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \ \vec{S}_F = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \tag{1.10}$$

the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geq 0$,
the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$

$$\tilde{I} := \frac{1}{4\pi} \int_{\mathcal{S}^2} I(\cdot, \vec{\omega}) \, \mathrm{d}\vec{\omega}$$

$$B(\nu, \vartheta) = 2h\nu^3 c^{-2} \left(e^{\frac{h\nu}{k\vartheta}} - 1 \right)^{-1}$$

– the radiative equilibrium function

h and k are the Planck and Boltzmann constants, the boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0, \tag{1.11}$$

$$I(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \ \vec{\omega} \cdot \vec{n} \leq 0, \tag{1.12}$$

$\vec{n}$ the outer normal vector to $\partial\Omega$.

- **Pomraning**
- **Mihalas and Weibel-Mihalas** in the framework of special relativity.
- astrophysics, laser applications (in the relativistic and inviscid case) by **Lowrie, Morel and Hittinger, Buet and Després**
- with a special attention to asymptotic regimes **Dubroca and Feugeas, Lin, Lin, Coulombel and Goudon**
  a simplified version of the system (non conducting fluid at rest) - investigated by **Golse and Perthame** , where global existence was proved by means of the theory of nonlinear semi-groups under very general hypotheses.

barotropic case

P. L. Lions (98)

$p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5$

generalization to a larger class of exponents $\gamma \geq 3/2$

E. Feireisl, A. Novotný and H. Petzeltová

Singular limits

P.L. Lions, B. Masmoudi

weak solution, Dirichlet conditions

Full system - the Navier - Stokes - Fourier system

global existence of weak solution E. Feireisl

Singular limits, Concept of weak- strong uniqueness

E. Feireisl, A. Novotný

Hypotheses

Pressure

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \; a > 0, \tag{2.1}$$

$P : [0, \infty) \to [0, \infty)$

$$P \in C^1[0, \infty), \; P(0) = 0, \; P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

$\frac{a}{3}\vartheta^4$- “equilibrium” radiation pressure.

the specific internal energy e is

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \qquad (2.5)$$

the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \qquad (2.6)$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

$$0 < c_1(1+\vartheta) \leq \mu(\vartheta),\ \ \mu'(\vartheta) < c_2,\ \ 0 \leq \eta(\vartheta) \leq c(1+\vartheta), \quad (2.7)$$

$$0 < c_1(1+\vartheta^3) \leq \kappa(\vartheta) \leq c_2(1+\vartheta^3) \quad (2.8)$$

for any $\vartheta \geq 0$.

$$0 \leq \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \leq c_1, \quad (2.9)$$

$$0 \leq \sigma_a(\nu,\vartheta)B(\nu,\vartheta), |\partial_\vartheta\{\sigma_a(\nu,\vartheta)B(\nu,\vartheta)\}| \leq c_2, \quad (2.10)$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \leq h(\nu),\ \ h \in L^1(0,\infty). \quad (2.11)$$

for all $\nu \geq 0$, $\vartheta \geq 0$, where $c_{1,2,3}$ are positive constants. **Relations (2.9) - (2.11) represent “cut-off” hypotheses neglecting the effect of radiation at large frequencies** ν

Weak formulation: *(weak) renormalized version represented by the family of integral identities*

$$\int_0^T \int_\Omega \Big(\Big(\varrho + b(\varrho)\Big)\partial_t\varphi + \Big(\varrho + b(\varrho)\Big)\vec{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\,\mathrm{d}t$$

(2.12)

$$+\int_0^T \int_\Omega \Big(\Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\varphi\Big)\ \mathrm{d}x\,\mathrm{d}t = -\int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot)\,\mathrm{d}x$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, where (2.12) implicitly includes the initial condition $\varrho(0,\cdot) = \varrho_0$.

The momentum equation (1.2) is replaced by

$$\int_0^T \int_\Omega \left(\varrho \vec{u} \cdot \partial_t \varphi + \varrho \vec{u} \otimes \vec{u} : \nabla_x \varphi + p \mathrm{div}_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.13}$$

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x \varphi + \vec{S}_F \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \vec{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$.

$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \tag{2.14}$$

where (2.14) already includes the no-slip boundary condition (1.11).

the entropy equation is replaced by the inequality

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho \vec{u} \cdot \nabla_x \varphi + \vec{q} \vartheta \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.15}$$

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \, \mathrm{d}x$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \frac{1}{\vartheta} \Big(\vec{u} \cdot \vec{S}_F - S_E \Big) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$, $\varphi \geq 0$.

where not the sign of all the terms in the right hand side may be controlled.

the total energy balance

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + E^R \right) (\tau, \cdot) \, \mathrm{d}x \tag{2.16}$$

$$+ \int_0^\tau \int \int_{\partial\Omega \times \mathcal{S}^2, \; \vec{\omega}\cdot\vec{n} \geq 0} \int_0^\infty \vec{\omega} \cdot \vec{n} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} \right) \, \mathrm{d}x,$$

where E^R is given by

$$E^R(t, x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu. \tag{2.17}$$

and $E_{R,0} = \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu.$

Definition 2.1 *We say that $\varrho, \vec{u}, \vartheta, I$ is a weak solution of problem (1.1 - 1.12) if*

$$\varrho \geq 0,\ \vartheta > 0 \text{ for a.a. } (t,x)\times\Omega,\ I \geq 0 \text{ a.a. in } (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty),$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)),$$

$$\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\ \vartheta \in L^2(0,T;W^{1,2}(\Omega)),$$

$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),\ I(t,\cdot) \in L^\infty(0,T;L^1(\Omega\times\mathcal{S}^2\times(0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, I satisfy the integral identities (2.12), (2.13), (2.15), (2.16), together with the transport equation (1.4).

Theorem

(B. Ducomet, E.Feireisl,Š. Nečasová) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.11).
Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.1 - 1.12) in the sense of Definition 2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{2.18}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon) + E_{R,\varepsilon} \right)(0,\cdot)\ \mathrm{d}x \tag{2.19}$$

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} \right)\ \mathrm{d}x \le E_0,$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, I\}$ is a weak solution of problem (1.1 - 1.12).

Asymptotic analysis

1.Simplified model $\vec{S}_F = 0$

Derivation of the entropy inequality

The internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\vec{u}) + \mathrm{div}_x\vec{q} = \mathbb{S} : \nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} - S_E. \quad (2.20)$$

the entropy equation

$$\partial_t\left(\varrho s\right) + \mathrm{div}_x\left(\varrho s\vec{u}\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \varsigma, \quad (2.21)$$

where

$$\varsigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) - \frac{S_E}{\vartheta}, \quad (2.22)$$

$\varsigma_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$ is the (positive) matter entropy production.

The formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\vec{\omega} d\nu, \quad (2.23)$$

$n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number.
Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \vec{\omega} \; d\vec{\omega} d\nu, \quad (2.24)$$

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \; S \; d\vec{\omega} d\nu =: \varsigma^R. \quad (2.25)$$

$$\partial_t \left(\varrho s + s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma + \varsigma^R. \quad (2.26)$$

The right-hand side of (2.25) rewrites

$$\varsigma^R =:$$

$$\frac{S_E}{\vartheta} - \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (B - I) \; d\vec{\omega} d\nu$$

$$- \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (\tilde{I} - I) \; d\vec{\omega} d\nu,$$

NSFRS system

$$\varepsilon\partial_t I + \vec{\omega}\cdot\nabla_x I = \sigma_a\left(B - I\right) + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I \,\mathrm{d}\vec{\omega} - I\right), \tag{2.27}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{2.28}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\varepsilon^2}\,\nabla_x p(\varrho,\vartheta) - \,\mathrm{div}_x\mathbb{S} = 0. \tag{2.29}$$

$$\partial_t\left(\varrho e + \varepsilon E^R\right) + \mathrm{div}_x\left(\varrho e\vec{u} + \vec{F}^R\right) + \mathrm{div}_x\vec{q} = \varepsilon^2\mathbb{S} : \nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} \tag{2.30}$$

$$\partial_t \left(\varrho s + \varepsilon s^R \right) + \operatorname{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \ \operatorname{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \varsigma_\varepsilon, \qquad (2.31)$$

with

$$\varsigma_\varepsilon = \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \ d\vec{\omega} d\nu$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \ d\vec{\omega} d\nu,$$

$$\frac{d}{dt}\int_{\Omega}\left(\frac{\varepsilon^2}{2}\,\varrho|\vec{u}|^2+\varrho e+\varepsilon E^R\right)\,dx+\int_0^{\infty}\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}I\,\,d\Gamma_+ d\nu=0 \tag{2.32}$$

where $\Gamma_+ = \{(x,\vec{\omega})\in\partial\Omega\times\mathcal{S}^2\ :\ \vec{\omega}\cdot\vec{n}_x>0\}$.

the Radiative-Oberbeck-Boussinesq system (ROB)

$$\mathrm{div}_x \vec{U} = 0, \tag{2.33}$$

$$\overline{\varrho}\left(\partial_t \vec{U} + \mathrm{div}_x(\vec{U} \otimes \vec{U})\right) + \nabla_x \Pi - 2\mathrm{div}_x\left(\overline{\mu}\ \mathbb{D}(\vec{U})\right) = 0, \tag{2.34}$$

$$\overline{\varrho}\ \overline{c_P}\left(\partial_t \Theta + \mathrm{div}_x\left(\Theta \vec{U}\right)\right) - \mathrm{div}_x\left(\overline{\kappa}\nabla\Theta\right)$$

$$= \left\{\int_0^\infty \partial_\vartheta\left(\sigma_a(\nu,\overline{\vartheta})B(\nu,\overline{\vartheta}) + \sigma_a(\nu,\overline{\vartheta})\partial_\vartheta B(\nu,\overline{\vartheta})\right)\ d\nu\right.$$

$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \partial_\vartheta \sigma_a(\nu,\overline{\vartheta}) I_0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu\right\}\Theta$$

$$- \int_0^\infty \int_{\mathcal{S}^2} \sigma_a(\nu, \overline{\vartheta}) I_1(x, \nu, \vec{\omega}) \; d\vec{\omega} \; d\nu. \tag{2.35}$$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_a(\nu, \overline{\vartheta}) \left(B(\nu, \overline{\vartheta}) - I_0 \right) + \sigma_s(\nu, \overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0 \, \mathrm{d}\vec{\omega} - I_0 \right), \tag{2.36}$$

$$\vec{\omega} \cdot \nabla_x I_1 = \left\{ \sigma_a(\nu, \overline{\vartheta}) \partial_\vartheta B(\nu, \overline{\vartheta}) + \partial_\vartheta \sigma_a(\nu, \overline{\vartheta}) B(\nu, \overline{\vartheta}) \right.$$
$$\left. + \partial_\vartheta \sigma_s(\nu, \overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0(x, \nu, \vec{\omega}') \, \mathrm{d}\vec{\omega}' - I_0(x, \nu, \vec{\omega}) \right) \right\} \Theta$$
$$- \sigma_a(\nu, \overline{\vartheta}) I_1(x, \nu, \vec{\omega})$$
$$+ \sigma_s(\nu, \overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1(x, \nu, \vec{\omega}') \, \mathrm{d}\vec{\omega}' - I_1(x, \nu, \vec{\omega}) \right). \tag{2.37}$$

We finally consider the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0, \ \nabla\Theta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{2.38}$$

for (4.30)-(4.32) and

$$I_0(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0 \tag{2.39}$$

$$I_1(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0 \tag{2.40}$$

for (4.33) and (2.37), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \Theta|_{t=0} = \Theta_0. \tag{2.41}$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.
For any $T > 0$ the initial-bounday value problem (4.30) and (2.41) has at least a weak solution $(\vec{U}, \Theta, I_0, I_1)$ such that

1.
$$\vec{U} \in L^\infty(0,T;\mathcal{H}(\Omega)) \cap L^2(0,T;\mathcal{V}(\Omega)),$$
with $\mathcal{H}(\Omega) = \{\vec{U} \in L^2(\Omega;\mathbb{R}^3),\ \mathrm{div}_x \vec{U} = 0 \text{ in } \Omega,\ \ \vec{U}\Big|_{\partial\Omega} = 0\}$,
and $\mathcal{V}(\Omega) = \mathcal{H}(\Omega) \cap W_0^{1,2}(\Omega;\mathbb{R}^3))$,

2.
$$\Theta \in V_2^{1,1/2}((0,T) \times \Omega),$$

3.
$$I_0, I_1 \in L^\infty((0,T) \times \Omega) \times \mathcal{S}^2 \times \mathbb{R}_+),$$

with

$$\vec{\omega} \cdot \nabla_x I_0, \ \vec{\omega} \cdot \nabla_x I_1 \in L^p((0,T) \times \Omega) \times \mathcal{S}^2 \times \mathbb{R}_+,$$

for any $p > 1$.

Let us choose initial data such that

$$\begin{cases} \varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \\ \vec{u}(0,\cdot) = \vec{u}_{0,\varepsilon}, \\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \\ I(0,\cdot,\cdot,\cdot) = I_{0,\varepsilon} = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon}, \end{cases} \tag{3.1}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\, dx = 0$ for any $\varepsilon > 0$.

- the existence of a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ to the radiative Navier-Stokes system (1.1 - 1.9)
- uniform estimates

Theorem

(B. Ducomet, Š.N.) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11).
Then $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (1.1 - 1.9) for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ be given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \mathbb{I}_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \overline{I} > 0,$$

are constants and

$$\int_\Omega \varrho_{0,\varepsilon}^{(1)}\ dx = 0,\ \int_\Omega \vartheta_{0,\varepsilon}^{(1)}\ dx = 0,\ \int_\Omega I_{0,\varepsilon}^{(1)}\ dx = 0 \ \textit{for all}\ \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho_{0,\varepsilon}^{(1)} \rightarrow \varrho_0^{(1)} \ \ weakly - (*) \ in \ L^\infty(\Omega), \\ \vec{u}_{0,\varepsilon}^{(1)} \rightarrow \vec{U}_0 \ \ weakly - (*) \ in \ L^\infty(\Omega; \mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \rightarrow \vartheta_0^{(1)} \ \ weakly - (*) \ in \ L^\infty(\Omega), \\ I_{0,\varepsilon}^{(1)} \rightarrow I_0^{(1)} \ \ weakly - (*) \ in \ L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+). \end{cases}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{4}{3}}(\Omega)} \leq C\varepsilon,$$

and up to subsequences

$$\vec{u}_\varepsilon \to \vec{U} \ \textit{weakly} - (*) \ \textit{in} \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta^{(1)} \to \Theta \ \ \textit{weakly} - (*) \ \textit{in} \ L^2(0,T;W^{1,2}(\Omega)),$$

$$I_\varepsilon \to I_0 \ \ \textit{weakly} - (*) \ \textit{in} \ L^2(0,T;L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

and

$$\frac{I_\varepsilon - \overline{I}}{\varepsilon} = I^{(1)} \to I_1 \quad \textit{weakly} - (*) \ \textit{in} \ L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

where $(\vec{U}, \Theta, I_0, I_1)$ *solve the radiative Oberbeck-Boussinesq system (4.30)-(2.37).*

Semi-relativistic model

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)} - 1}, \tag{4.1}$$

$0 \leq \alpha(\vartheta) \leq 1$

If $\frac{|\vec{u}|}{c} << 1$ one recovers the standard equilibrium Planck's function $B(\nu, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}} - 1}$.

Berthon, Buet, Coulombel, Depres, Dubois, Goudon, Morel, Turpault

$M1$ Levermore model

$$\alpha = \frac{\sigma_a + \sigma_s}{\sigma_a + 2\sigma_s}, \tag{4.2}$$

$\sigma_a(\vartheta, \vec{u}) = \chi(|\vec{u}|)\tilde{\sigma}_a(\vartheta) \geq 0$ and $\sigma_s(\vartheta) \geq 0$

$$\chi(s) = \begin{cases} 1 & \text{if } \; s \leq c, \\ 0 & \text{if } \; s \geq c + \beta, \end{cases}$$

for an arbitrary $\beta > 0$.

The role of this cut-off is to deal with the singularity of B

In the "over-relativistic" regime ($|\vec{u}| \geq c$) we decide to decouple matter and radiation.

Scaled system

$$\varepsilon\partial_t I + \vec{\omega}\cdot\nabla_x I = \sigma_a (B - I) + \sigma_s \left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I \,\mathrm{d}\vec{\omega} - I \right), \qquad (4.3)$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \qquad (4.4)$$

$$\partial_t \left(\varrho\vec{u} + \varepsilon^2 \vec{F}^R\right) + \mathrm{div}_x \left(\varrho\vec{u}\otimes\vec{u} + \varepsilon\mathbb{P}^R\right) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) - \mathrm{div}_x\mathbb{S} = 0. \qquad (4.5)$$

$$\partial_t \left(\frac{\varepsilon^2}{2} \varrho |\vec{u}|^2 + \varrho e + \varepsilon E^R \right) + \tag{4.6}$$

$$+\mathrm{div}_x \left(\left(\frac{\varepsilon^2}{2} \varrho |\vec{u}|^2 + \varrho e + p \right) \vec{u} + \vec{F}^R + \vec{q} - \varepsilon^2 \mathbb{S} \vec{u} \right) = 0,$$

$$\partial_t \left(\varrho s + \varepsilon s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq$$

$$\geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \ d\vec{\omega} d\nu$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \ d\vec{\omega} d\nu =: \varsigma_\varepsilon, \tag{4.7}$$

with the conservation of total energy

$$\frac{d}{dt}\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + \varepsilon E^R\right)\ dx + \int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n} I\ d\Gamma_+ d\nu = 0. \tag{4.8}$$

the renormalized radiative energy, $\vec{F}^R = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega} I \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega}$,

the renormalized radiative momentum, by

$\mathbb{P}^R = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\otimes\vec{\omega} I \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega}$,

Limit system

$$\mathrm{div}_x \vec{U} = 0, \qquad (4.9)$$

$$\overline{\varrho} \left(\partial_t \vec{U} + \mathrm{div}_x (\vec{U} \otimes \vec{U}) \right) + \nabla_x \Pi - 2\mathrm{div}_x \left(\overline{\mu}\, \mathbb{D}(\vec{U}) \right) = 0, \qquad (4.10)$$

$$\overline{\varrho}\ \overline{c_P}\left(\partial_t\Theta + \mathrm{div}_x\left(\Theta\vec{U}\right)\right) - \mathrm{div}_x\left(\overline{\kappa}\nabla\Theta\right)$$

$$= \{\int_0^\infty \partial_\vartheta\left(\sigma_a(\overline{\vartheta})B(\nu,\overline{\vartheta}) + \sigma_a(\overline{\vartheta})\partial_\vartheta B(\nu,\overline{\vartheta})\right)\ d\nu +$$

$$\int_0^\infty \int_{\mathcal{S}^2} \partial_\vartheta\sigma_a(\overline{\vartheta})I_0\ d\vec{\omega}\ d\nu\Theta - \int_0^\infty \int_{\mathcal{S}^2} \sigma_a(\overline{\vartheta})I_1\ d\vec{\omega}\ d\nu,$$

$$\vec{\omega}\cdot\nabla_x I_0 = \sigma_a(\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right) + \sigma_s(\overline{\vartheta})\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_0 \,\mathrm{d}\vec{\omega} - I_0\right), \tag{4.12}$$

$$\vec{\omega}\cdot\nabla_x I_1 = \Big\{\sigma_a(\overline{\vartheta})\partial_\vartheta B(\nu,\overline{\vartheta}) + \partial_\vartheta\sigma_a\overline{\vartheta})B(\nu,\overline{\vartheta})$$

$$+\partial_\vartheta\sigma_s(\overline{\vartheta})\left(\tfrac{1}{4\pi}\textstyle\int_{\mathcal{S}^2} I_0(x,\nu,\vec{\omega}')\,\mathrm{d}\vec{\omega}' - I_0(x,\nu,\vec{\omega})\right)\Big\}\Theta$$

$$-\sigma_a(\overline{\vartheta})I_1(x,\nu,\vec{\omega}) + \sigma_s(\overline{\vartheta})\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1(x,\nu,\vec{\omega}')\,\mathrm{d}\vec{\omega}' - I_1(x,\nu,\vec{\omega})\right). \tag{4.13}$$

We finally consider the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0, \;\; \nabla\Theta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{4.14}$$

for (4.9)-(4.11) and

$$I_0(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \;\; \vec{\omega}\cdot\vec{n} \le 0 \tag{4.15}$$

$$I_1(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \;\; \vec{\omega}\cdot\vec{n} \le 0 \tag{4.16}$$

for (4.12) and (4.13), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \Theta|_{t=0} = \Theta_0, \tag{4.17}$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.
For any $T > 0$ the initial-bounday value problem (4.30) - (2.41) has at least a weak solution $(\vec{U}, \Theta, I_0, I_1)$ such that

$$\vec{U} \in L^\infty(0, T; \mathcal{H}(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega)),$$

with $\mathcal{H}(\Omega) = \{\vec{U} \in L^2(\Omega; \mathbb{R}^3),\ \mathrm{div}_x \vec{U} = 0$ *in* $\Omega,\ \vec{U}\Big|_{\partial\Omega} = 0\}$, *and* $\mathcal{V}(\Omega) = \mathcal{H}(\Omega) \cap W_0^{1,2}(\Omega; \mathbb{R}^3))$,

$$\Theta \in V_2^{1,1/2}((0, T) \times \Omega),$$

$$I_0, I_1 \in L^\infty((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+),$$

with

$$\vec{\omega}\cdot\nabla_x I_0,\ \vec{\omega}\cdot\nabla_x I_0 \in L^p((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+),$$

for any $p > 1$.

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11).

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (1.1 - 1.9) for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ be given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \mathbb{I}_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \overline{I} > 0,$$

are constants and

$$\int_\Omega \varrho_{0,\varepsilon}^{(1)}\ dx = 0,\ \int_\Omega \vartheta_{0,\varepsilon}^{(1)}\ dx = 0,\ \int_\Omega I_{0,\varepsilon}^{(1)}\ dx = 0 \text{ for all } \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho_{0,\varepsilon}^{(1)} \longrightarrow \varrho_0^{(1)} \;\; weakly - (*) \;\; in \;\; L^\infty(\Omega), \\ \vec{u}_{0,\varepsilon}^{(1)} \longrightarrow \vec{U}_0 \;\; weakly - (*) \;\; in \;\; L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \longrightarrow \vartheta_0^{(1)} \;\; weakly - (*) \;\; in \;\; L^\infty(\Omega), \\ I_{0,\varepsilon}^{(1)} \longrightarrow I_0^{(1)} \;\; weakly - (*) \;\; in \;\; L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+). \end{cases}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{4}{3}}(\Omega)} \leq C\varepsilon,$$

and up to subsequences

$$\vec{u}_\varepsilon \to \vec{U} \ weakly - (*) \ in \ L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)),$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta^{(1)} \to \Theta \ \ weakly - (*) \ in \ L^2(0, T; W^{1,2}(\Omega)),$$

$$I_\varepsilon \to I_0 \ \ weakly - (*) \ in \ L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

and

$$\frac{I_\varepsilon - \overline{I}}{\varepsilon} = I^{(1)} \to I_1 \ \ weakly - (*) \ in \ L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

where $(\vec{U}, \Theta, I_0, I_1)$ solve the radiative Oberbeck-Boussinesq system.

Diffusion limit

equilibrium diffusion regime

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = \frac{1}{\varepsilon}\sigma_a\,(B - I) + \varepsilon\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \quad (4.18)$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \quad (4.19)$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) - \ \mathrm{div}_x\mathbb{T} = 0. \quad (4.20)$$

$$\partial_t\left(\varrho s+\varepsilon s_R\right)+\operatorname{div}_x\left(\varrho\vec{u}s+\vec{q}_R\right)+\operatorname{div}_x\left(\frac{\vec{q}}{\vartheta}\right)\geq\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u}-\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

$$+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$

$$+\varepsilon\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega}d\nu, \tag{4.21}$$

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\ \varrho|\vec{u}|^2+\varrho e+E_R\right)\ dx+\frac{1}{\varepsilon}\int_0^\infty\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}\ I\ d\Gamma_+d\nu=0. \tag{4.22}$$

the " non-equilibrium diffusion regime"

$$\varepsilon\, \partial_t I + \vec{\omega} \cdot \nabla_x I = \varepsilon \sigma_a \,(B - I) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \, \mathrm{d}\vec{\omega} - I \right), \quad (4.23)$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \quad (4.24)$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) - \mathrm{div}_x \mathbb{T} = 0. \quad (4.25)$$

$$\partial_t \left(\varrho s + \varepsilon s_R\right) + \operatorname{div}_x \left(\varrho \vec{u} s + \vec{q}_R\right) + \operatorname{div}_x \left(\frac{\vec{q}}{\vartheta}\right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

$$+\varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1}\right] \sigma_a(I - B)\ d\vec{\omega} d\nu$$

$$+ \frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1}\right] \sigma_s(I - \tilde{I})\ d\vec{\omega} d\nu. \tag{4.26}$$

The equilibrium-diffusion regime

$$\begin{cases} I = I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + O(\varepsilon^3), \\ \varrho = \rho_0 + \varepsilon \varrho_1 + \varepsilon^2 \varrho_2 + O(\varepsilon^3), \\ \vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + \varepsilon^2 \vec{u}_2 + O(\varepsilon^3), \\ \vartheta = \vartheta_0 + \varepsilon \vartheta_1 + \varepsilon^2 \vartheta_2 + O(\varepsilon^3). \end{cases} \tag{4.27}$$

$$I_0 = B(\nu, \vartheta_0), \tag{4.28}$$

$$I_1 = \partial_\vartheta B(\nu, \vartheta_0)\vartheta_1 - \frac{1}{\sigma_a(\vartheta_0)}\, \vec{\omega} \cdot \nabla_x I_0. \tag{4.29}$$

The limit system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{4.30}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta), \tag{4.31}$$

$$\partial_t \left(\varrho \mathcal{E}(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta)\vec{u}\right) + \mathrm{div}_x \left(\mathcal{K}(\varrho, \vartheta)\nabla_x \vartheta\right)$$

$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \vec{u} - p(\varrho, \vartheta)\mathrm{div}_x \vec{u}, \tag{4.32}$$

$$I = B(\nu, \vartheta). \tag{4.33}$$

$$\vec{u}|_{\partial\Omega} = 0, \ \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{4.34}$$

$$(\varrho(x,t), \ \ \vec{u}(x,t), \ \ \vartheta(x,t))|_{t=0} = \left(\varrho^0(x), \ \ \vec{u}^0(x), \ \ \vartheta^0(x)\right), \tag{4.35}$$

$$\mathcal{E}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{B(\vartheta)}{\varrho}, \text{ and } \mathcal{K}(\vartheta) = \kappa(\vartheta) - \frac{1}{3\sigma_a(\vartheta)}\,\partial_\vartheta B(\vartheta).$$

Matsumura and Nishida- a global existence result of strong solution for this system, for small data.

Remark:

When one considers the formal "nonconducting at rest" situation where $\kappa = 0$ and $\vec{u} = 0$ and in the no-scattering case ($\sigma_s \equiv 0$), one obtains the simplified system introduced by Bardos, Golse and Perthame for which they proved global existence and diffusion limit (called "Rosseland approximation") under assumptions much more general than ours.

The non-equilibrium diffusion regime

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{4.36}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta), \tag{4.37}$$

$$\partial_t \left(\varrho e(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta)\vec{u}\right) + \mathrm{div}_x \left(\kappa(\vartheta)\nabla_x \vartheta\right)$$

$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \vec{u} - p(\varrho, \vartheta)\mathrm{div}_x \vec{u} - \sigma_a(\vartheta)[B(\vartheta) - N], \tag{4.38}$$

$$\partial_t N - \frac{1}{3}\, \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta)} \nabla_x N \right) = \sigma_a(\vartheta) \left(B(\vartheta) - N\right). \tag{4.39}$$

$$\vec{u}|_{\partial\Omega} = 0,\ \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{4.40}$$

$$N := \int_0^\infty I_0 \; d\nu$$

$$N|_{\partial\Omega} = 0. \tag{4.41}$$

$$\left(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t), N(x,t)\right)|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x), N^0(x)\right), \tag{4.42}$$

$N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu.$

$(\overline{\varrho}, 0, \overline{\vartheta}, \overline{N})$ be a given constant state with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{N} = B(\overline{\vartheta})$.

$$e_0 := \|\varrho^0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vec{u}^0\|_{H^1(\Omega)} + \|\vartheta^0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|N^0 - \overline{N}\|_{H^1(\Omega)} + \|\mathbb{T}^0\|_{L^2(\Omega)} + \|\mathbb{V}^0\|_{L^4(\Omega)}, \tag{4.43}$$

$$E_0 := e_0 + \|\nabla_x \varrho^0\|_{L^2(\Omega)} + \|\nabla_x \varrho^0\|_{L^\alpha(\Omega)} + \|\nabla_x \mathbb{T}^0\|_{L^2(\Omega)}, \tag{4.44}$$

$3 < \alpha < 6$.

Strong solution for small data

Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}, \qquad (4.45)$$

$\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \ : \ \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, \qquad (4.46)$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}. \qquad (4.47)$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.4) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11), $B \in C^1$.

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$

such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta_{0,\varepsilon}^{(1)},$$

where $(\varrho_0, \vec{u}, \vartheta_0) \in H^3(\Omega)$ are smooth functions such that (ϱ_0, ϑ_0) belong to the set $\mathcal{O}_{ess}^H$ defined in (4.47) where $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, are two constants and $\int_\Omega \varrho_{0,\varepsilon}^{(1)}\, dx = 0$, $\int_\Omega \vartheta_{0,\varepsilon}^{(1)}\, dx = 0$.

Suppose also that

$$\vec{u}_{0,\varepsilon} \to \vec{u}_0 \;\; strongly \;\; in \;\; L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \;\; strongly \;\; in \;\; L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \;\; strongly \;\; in \;\; L^2(\Omega),$$

$$I_{0,\varepsilon}^{(1)} \to I_0^{(1)} \;\; strongly \;\; in \;\; L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \;\; strongly \;\; in \;\; L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \;\; strongly \;\; in \;\; L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \;\; strongly \;\; in \;\; L^\infty(0,T;L^4(\Omega)),$$

where $(\varrho,\vec{u},\vartheta)$ is the smooth solution of the equilibrium decoupled system (4.30)-(4.33) on $[0,T]\times\Omega$ and $I(t,x,\nu,\vec{\omega}) = B(\nu,\vartheta(t,x))$, with initial data $(\varrho_0,\vec{u}_0,\vartheta_0)$.

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.4) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11), $B \in C^1$.
Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.11 - 1.12) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

$$I_\varepsilon(0,\cdot) = I_0 + \varepsilon I^{(1)}_{0,\varepsilon},$$

where the functions $(\varrho_0, \vec{u}, \vartheta_0)$ and $x \longrightarrow I_0(x, \vec{\omega}, \nu)$ belong to $H^3(\Omega)$ and are such that $(\varrho_0, \vartheta_0, E_R(I_0))$ belong to the set $\mathcal{O}_{ess}$ defined in (4.47) where $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, $\overline{E_R} > 0$ are three constants and $\int_\Omega \varrho_{0,\varepsilon}^{(1)}\, dx = 0$, $\int_\Omega \vartheta_{0,\varepsilon}^{(1)}\, dx = 0$, $\int_\Omega I_{0,\varepsilon}^{(1)}\, dx = 0$.

Suppose also that

$$\vec{u}_{0,\varepsilon} \longrightarrow \vec{u}_0 \ \ \textit{strongly in } L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho^{(1)}_{0,\varepsilon} \longrightarrow \varrho^{(1)}_0 \ \textit{strongly in } L^2(\Omega),$$

$$\vartheta^{(1)}_{0,\varepsilon} \longrightarrow \vartheta^{(1)}_0 \ \textit{strongly in } L^2(\Omega),$$

$$I^{(1)}_{0,\varepsilon} \longrightarrow I^{(1)}_0 \ \textit{strongly in } L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \longrightarrow \varrho \ \ \textit{strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \longrightarrow \vec{u} \ \textit{strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \longrightarrow \vartheta \ \ \textit{strongly in } L^\infty(0,T;L^4(\Omega)),$$

and

$$N_\varepsilon \to N \ \ strongly \ \ in \ L^\infty((0,T) \times \Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \ d\vec{\omega} \ \ d\nu$ and $(\varrho, \vec{u}, \vartheta, N)$ is the smooth solution of the Navier-Stokes-Rosseland system (4.36)-(4.39) on $[0, T] \times \Omega$ with initial data $(\varrho_0, \vec{u}_0, \vartheta_0, N_0)$.

We establish a relative entropy inequality satisfied by any weak solution $(\varrho, \vec{u}, \vartheta, I)$ *of the radiative Navier-Stokes system*

Let us consider a set $\{r, \Theta, \vec{U}\}$ of smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$. Notation:

We call *ballistic free energy* the thermodynamical potential given by

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta),$$

and *radiative ballistic free energy* the potential

$$H^R_\Theta(I) = E^R(I) - \Theta s^R(I).$$

The *relative entropy* is then defined by

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) := H_\Theta(\varrho, \vartheta) - \partial_\varrho H_\Theta(r, \Theta)(\varrho - \Theta) - H_\Theta(r, \Theta).$$

$$\int_\Omega \left(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) \right)(\tau, \cdot) dx$$

$$+ \int_0^\tau \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t, x, \vec{\omega}, \nu)\ d\Gamma d\nu dt$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right) dxdt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon) + 1} \right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu dx$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon) + 1} \right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu dx d$$

$$\leq -\int_\Omega \frac{1}{2}\left(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}|r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon})\right)$$

$+$

$$\int_0^\tau \int_\Omega R(x,t)$$

$(r = \rho, \vec{U} = \vec{u}, \Theta = \vartheta)$, where $(\varrho, \vec{u}, \vartheta)$ is a classical solution of the target system (in the equilibrium case or in the non equilibrium case)

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{u}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta\right) \right)\ dx \tag{4.48}$$

$$\leq \left(\int_\Omega \left(\frac{1}{2} \left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{u}(0,\cdot)|^2 + \right.\right.\right.$$

$$\left.\left.\left. \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | \varrho(0,\cdot), \theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \right) dx + (K_4 e_0(\epsilon + 1)) e^{K_3 t}.\right.\right.$$

Remark:

- In the case of semi-relativistic case we prove the limit as before but without smallness of e_0
- non-relativistic limit