# The Taylor model in magnetohydrodynamics (MHD)

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# 1. Motivations

Main issue: To understand the Earth's magnetic field.

Consensus: due to dynamo effect.

Dynamo: Conversion of mechanical energy into magnetic energy.

Fluid dynamo : The mechanical energy is the kinetic energy of a conducting fluid.

Example: Liquid iron in the Earth's core.

Starting point: MHD equations.

Comes from a simplification of the coupling between (incompressible) Navier-Stokes and Maxwell

$u$ : fluid velocity field, $p$ : pressure field, $B$ : magnetic field.

We account for the *Coriolis forcing* due to rotation (with axis $e = e_z$). The centrifugal force is included in the pressure gradient.

$$
\boxed{\begin{aligned}
&\rho(\partial_t u + u\cdot\nabla u) + \nabla p + \rho\Omega_0 e\times u - \mu\Delta u = \mu_0^{-1}\,\mathrm{curl}\, B\times B\\
&\partial_t B = \mathrm{curl}\,(u\times B) + \eta\Delta B,\\
&\mathrm{div}\, u = \mathrm{div}\, B = 0.
\end{aligned}}
\tag{mhd}
$$

- $\mathrm{curl}\,(u\times B)$ : induction, related to Lorentz force.
- $\mathrm{curl}\, B\times B$ : Laplace force.

Remark: The constraint $\mathrm{div}\, B = 0$ is preserved by the evolution

Many mathematical works about dynamo Arnold, Vishik, Friedlander, Childress, Gilbert... Notably related to the single equation

$$
\boxed{\partial_t B = \mathrm{curl}\,(u\times B) + \eta\Delta B, \quad u \text{ fixed.}}
$$

Question: What fields $u$ generate exponentially growing $B$ ?

Idea: $u$ should be "complex" (no symmetries, lagrangian chaos).

$\rightarrow$ fine variations of $u$ must be captured in the numerics.

Problem: A lot of small parameters !

In dimensionless form:

$$\boxed{\begin{aligned} &\partial_t u + u\cdot\nabla u + \frac{\nabla p}{\varepsilon} + \frac{e\times u}{\varepsilon} - \frac{E}{\varepsilon}\Delta u = \frac{\Lambda}{\varepsilon\theta}\operatorname{curl} B\times B \\ &\partial_t B = \operatorname{curl}(u\times B) + \frac{1}{\theta}\Delta B, \\ &\operatorname{div} u = \operatorname{div} B = 0. \end{aligned}} \qquad \text{(MHD)}$$

$\varepsilon, E, \Lambda, \theta$ : Rossby, Ekman, Elsasser and magnetic Reynolds numbers.

Example : Earth :

$$\varepsilon \sim 10^{-7}, \quad \Lambda = O(1), \quad \varepsilon\theta \sim 10^{-4}, \quad E \sim 10^{-15}.$$

Formal limit model : Taylor'1963

$$
\boxed{
\begin{aligned}
& e \times u + \nabla p = \frac{\Lambda}{\theta} \operatorname{curl} B \times B, \quad \operatorname{div} u = 0 \\
& \partial_t B = \operatorname{curl}(u \times B) + \frac{1}{\theta} \Delta B, \quad \operatorname{div} B = 0.
\end{aligned}
}
\tag{T}
$$

Remarks:

- Singular perturbation problem
- Nonlinear penalization.

Question : Relevance of the limit system ?

# 3. Study of the Taylor model

**Step 1 :** Equations on $u$, for a given $B$ ($\Lambda = \theta = 1$) :

$$e \times u + \nabla p = \operatorname{curl} B \times B, \quad \operatorname{div} u = 0 \quad \text{in } \Omega,$$

with $u \cdot n|_{\partial\Omega} = 0$. Can be written

$$\boxed{\mathbb{C}u = \mathbb{P}\left(\operatorname{curl} B \times B\right)}$$

$\mathbb{P}$ : Leray projector, $\mathbb{C} = \mathbb{P}(e \times \cdot)$ : Coriolis operator.

Remark: $\mathbb{C}$ skew-symmetric over

$$L^2_\sigma := \left\{u \in L^2(\Omega)^3, \operatorname{div} u = 0, u \cdot n|_{\partial\Omega} = 0\right\}.$$

This equation has a solution under the *Taylor constraint*:

$$\boxed{\mathbb{P}\left(\operatorname{curl} B \times B\right) \in \mathrm{Range}(\mathbb{C})}$$

Defines a solution $u$ up to an element of $\ker(\mathbb{C})$ : $\boxed{u = u_m + u_g}$

with $u_m \in \ker(\mathbb{C})^\perp$ and $u_g \in \ker(\mathbb{C})$ (magneto- and geostrophic).

**Step 2 :** Equation on $B$ :

$$\boxed{\partial_t B = \operatorname{curl}(u_m \times B) + \operatorname{curl}(u_g \times B) + \Delta B = 0.}$$

*The term curl$(u_g \times B)$ is a kind of Lagrange multiplier, associated to the Taylor constraint.*

<u>Remark</u>: Navier-Stokes:

- $\nabla p$ : multiplier associated to the constraint $\operatorname{div} u = 0$.
- This constraint is preserved iff $-\Delta p = \operatorname{div}(u \cdot \nabla u)$ in $\Omega$ (with a Neumann condition on $p$ at $\partial\Omega$).

Problem: Is there an equation on $u_g$ ?

Unclear ! Depends on

- the geometry of $\Omega$.
- the elements $u_g$ in $\ker(\mathbb{C})$.

Example 1: $\Omega = \mathbb{T}^3$,

$$u_g = u_g(x_1, x_2), \text{ with } \partial_1 u_{g,1} + \partial_2 u_{g,2} = 0.$$

Example 2 (more realistic): $\Omega = B(0,1)$,

$$u_g = (0, U_\theta(r), 0)$$

in cylindrical coordinates.

In this case, Taylor derives formally an elliptic equation for $u_\theta$.

## 4. Modified Taylor model (without curvature)

In $\Omega = \mathbb{T}^3$,

$$\begin{cases} e \times u + \nabla p = \operatorname{curl} B \times B, \quad \operatorname{div} u = 0, \\ \partial_t B = \operatorname{curl}(u \times B) + \Delta B, \quad \operatorname{div} B = 0, \\ \displaystyle\int_{\mathbb{T}} u_1 dz = \int_{\mathbb{T}} u_3 dz = 0. \end{cases} \tag{T2}$$

One applies curl to the first line : $\partial_z u = \operatorname{curl}(\operatorname{curl} B \times B)$.

*Taylor constraint* reads

$$\int_{\mathbb{T}} \operatorname{curl}(\operatorname{curl} B \times B) = 0, \quad \text{thus} \quad u_m = \partial_z^{-1} \operatorname{curl}(\operatorname{curl} B \times B)$$

(antiderivative with zero mean). Elements of $\ker(\mathbb{C})$ take the form

$$u_g = (0, U(x_1), 0).$$

Using the 2nd equation in the relation $\partial_t \int_{\mathbb{T}} \text{curl}\,(\text{curl}\, B \times B) = 0$ leads to

$$\boxed{\frac{d}{dx_1}\left(\alpha(x_1)\frac{d}{dx_1}U\right) \;=\; \frac{d}{dx_1}\mathcal{F}(u_m, B)}$$

with $\alpha(x_1) = \int_{\mathbb{T}^2} B_1^2(x_1, x_2, x_3)dx_2\, dx_3$.

If $\alpha > 0$, the model reduces to an evolution equation on $B$.

**Main question : Well-posedness of the Cauchy problem ?**

Seems a hard question. We restrict to some linearized problem, with some smooth reference solution $\overline{B}$.

$$\boxed{\begin{cases} e \times u + \nabla p = \text{curl}\, b \times \overline{B} + \text{curl}\, \overline{B} \times b, \quad \text{div}\, u = 0, \\ \partial_t b = \text{curl}\,(u \times \overline{B}) + \text{curl}\,(\overline{u} \times b) + \Delta b, \quad \text{div}\, b = 0, \\ \displaystyle\int_{\mathbb{T}} u_1 dz \;=\; \int_{\mathbb{T}} u_3 dz = 0. \end{cases}} \qquad \text{(LT)}$$

Again : $u = u_m + u_g, \quad u_m = \partial_z^{-1}\text{curl}\,\big(\text{curl}\, b \times \overline{B} + \text{curl}\, \overline{B} \times b\big).$

Idea: focus on the $m$-term.

We consider some $\overline{B}$ such that the linearized Taylor constraint is preserved by the evolution

$\to u_g = 0.$

Example 1:

- $\overline{B} = \overline{B} = B(x_1, x_2)$.
- initial data $b_0$ of zero mean.

Example 2:

- $\overline{B} = (B(z), 0, 0),\ B > 0$.
- $B$ has Fourier modes with even indices, $b_0$ has Fourier modes with odd indices.

# Our main result is:

Theorem: The linearized system (LT) is well-posed in $L^2$ for these examples :

$$\boxed{||b(t)||_{L^2} \ \leq \ e^{Ct}\|b(0)\|_{L^2}}$$

# Our main result is:

Theorem: The linearized system (LT) is well-posed in $L^2$ for these examples :

$$\boxed{||b(t)||_{L^2} \;\leq\; e^{Ct}\|b(0)\|_{L^2}}$$

A few words on the proof (example 2)

Issue: the operator $b \mapsto \operatorname{curl}(u_m \times b)$, where

$$u_m = u_m(b) = \partial_z^{-1}\partial_z^{-1}\operatorname{curl}\,\left(\operatorname{curl} b \times \overline{B} + \operatorname{curl} \overline{B} \times b\right).$$

- Operator of order 3 in $x_1, x_2$ (and of order 2 in $z$).
- The leading part of the operator is skew-symmetric.
- For $\overline{B}$ large enough, derivatives of order $\leq 2$ could generate a growth like $e^{c(\xi_1^2+\xi_2^2)t}$.

To solve this issue, several steps are needed.

- Step 1.

Fourier : $(x_1, x_2) \to (\xi_1, \xi_2)$. Simple arguments allow to reduce to the regime:

$$\boxed{\xi_i = \frac{\eta_i}{\varepsilon}, \quad \varepsilon \ll 1, \quad \eta_1^2 + \eta_2^2 = 1, \quad 0 < \delta \leq \frac{\eta_1}{\eta_2} \leq \frac{1}{\delta}.}$$

In this regime, the linearized equation takes the form

$$\boxed{\partial_t b + \frac{\mathcal{A} b}{\varepsilon^3} + \frac{\mathcal{L} b}{\varepsilon^2} - \frac{b}{\varepsilon^2} + \cdots = 0}$$

where $\mathcal{A} b = -\eta_1^2 B \partial_z^{-1} B \begin{pmatrix} i\eta_1 \\ i\eta_2 \\ 0 \end{pmatrix} \times b, \quad \mathcal{L} b = \eta_1 \eta_2 B \partial_z^{-1} B' b.$

- Step 2.

Spectral analysis of operator $B\partial_z^{-1}B$. Explicit orthonormal basis of eigenvectors $e_k$, $k \in \mathbb{Z}$. The spectral analysis of $\mathcal{A}$ follows.

- Step 3.

Treatment of "high frequencies" : $\Pi^N b = \sum_{|k|\geq N} \begin{pmatrix} (b_1|e_k) \\ (b_2|e_k) \\ (b_3|e_k) \end{pmatrix} e_k$

One can conclude using that $\mathcal{L}$ contains a $\partial_z^{-1}$.

<u>Remark</u>: one shows that

High frequencies in Fourier $\approx$ high frequencies in the $e_k$'s (stationary phase theorem).

- Step 4.

Treatment of "low frequencies" : $\Pi_N b = \sum_{|k|<N} \begin{pmatrix} (b_1|e_k) \\ (b_2|e_k) \\ (b_3|e_k) \end{pmatrix} e_k.$

*Use of a normal form*: broadly, change of unknown of the form

$$\boxed{b_N^\varepsilon = (I + \varepsilon Q)\,\Pi_N b}.$$

Equation of the type

$$\boxed{\partial_t b_N^\varepsilon + \frac{\mathcal{A} b_N^\varepsilon}{\varepsilon^3} + \frac{([Q,\mathcal{A}] + \mathcal{L}) b_N^\varepsilon}{\varepsilon^2} - \frac{b_N^\varepsilon}{\varepsilon^2} + \cdots = 0}$$

One constructs $Q$ such that $[Q, \mathcal{A}] = -\mathcal{L}$.

<u>Remark</u> : potential trouble due to multiple eigenvalues.

- Eigenvalue 0 : pb connected to $\ker(\mathcal{A})$ is evacuated thanks to $\operatorname{div} b = 0$. *Almost orthogonality* (lower order in $\varepsilon$).
- Other eigenvalues have multiplicity 2. OK thanks to a compatibility condition satisfied by $\mathcal{L}$.