The DDFV method for the Stokes problem

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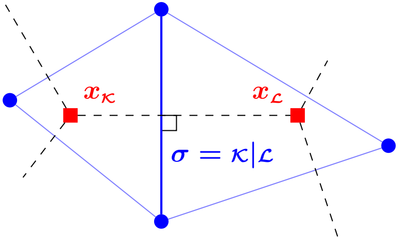
Outline

1. Introduction : two-point flux approximations
2. The DDFV method
3. Conclusions

Consider the following problem

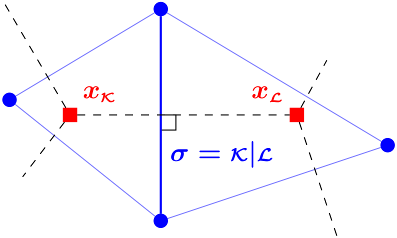
$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

and an admissible orthogonal mesh $\mathcal{T}$



FLUX BALANCE EQUATION ON THE CONTROL VOLUME $\mathcal{K}$

$$|\mathcal{K}| f_\mathcal{K} \stackrel{\text{def}}{=} \int_\mathcal{K} f = \int_\mathcal{K} -\Delta u = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \underbrace{-\int_\sigma \nabla u \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma}}_{\stackrel{\text{def}}{=} \overline{F}_{\mathcal{K},\sigma}(u)}$$

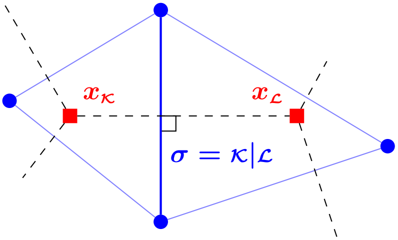


$$|\mathcal{K}| f_\mathcal{K} = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \overline{F}_{\mathcal{K},\sigma}(u).$$

CELL-CENTERED UNKNOWNS

We are looking for $u_\mathcal{K} \sim u(x_\mathcal{K})$

Notation : $u^\mathcal{T} = (u_\mathcal{K})_{\mathcal{K} \in \mathcal{T}} \in \mathbb{R}^\mathcal{T}$.



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Notation : $u^\mathcal{T} = (u_\mathcal{K})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^\mathcal{T}$.

CASE OF AN INTERIOR EDGE — $\sigma \in \mathcal{E}_{int}$, $\sigma = \mathcal{K}|\mathcal{L}$.

$$\overline{F}_{\mathcal{K},\sigma}(u) = \int_\sigma -(\nabla u)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = -|\sigma|\frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} + O(\text{size}(\mathcal{T})^2)$$

$$\boxed{F_{\mathcal{K},\sigma}(u^\mathcal{T}) \stackrel{\text{def}}{=} -|\sigma|\frac{u_\mathcal{L} - u_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}}.}$$

CASE OF A BOUNDARY EDGE — $\sigma \in \mathcal{E}_{ext}$.

$$\boxed{F_{\mathcal{K},\sigma}(u^\mathcal{T}) \stackrel{\text{def}}{=} -|\sigma|\frac{-u_\mathcal{K}}{d_{\mathcal{K}\sigma}}.}$$

Definition of the TPFA scheme

We look for $u^{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = |\mathcal{K}| f_{\mathcal{K}}, & \forall \mathcal{K}\in\mathcal{T}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma| \dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}, & \text{for } \sigma = \mathcal{K}|\mathcal{L} \in \mathcal{E}_{int}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma| \dfrac{-u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}}, & \text{for } \sigma \in \mathcal{E}_{ext}. \end{cases} \qquad \text{(TPFA)}$$

- It is a linear system of N equations with N unknowns (N=nb of control volumes in $\mathcal{T}$) which is invertible.
- The scheme is also known as VF4/FV4 : 4-point stencil for a triangle 2D mesh.
- On a 2D uniform Cartesian mesh : we recover the usual 5-point scheme.
- We can show that the scheme converges, we can obtain error estimates...

Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \\ \operatorname{div}(\mathbf{u}) &= 0, \quad \text{in } \Omega \\ \mathbf{u} &= 0, \quad \text{on } \partial\Omega \\ m(p) &= \frac{1}{|\Omega|}\int_\Omega p(x)\,\mathrm{d}x = 0, \end{cases}$$

FLUX BALANCE EQUATION ON THE CONTROL VOLUME $\mathcal{K}$

$$|\mathcal{K}|\mathbf{f}_\mathcal{K} \stackrel{\text{def}}{=} \int_\mathcal{K} \mathbf{f} = \int_\mathcal{K} (-\Delta \mathbf{u} + \nabla p) = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \underbrace{\int_\sigma (-\nabla \mathbf{u} \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma} + p\boldsymbol{\nu}_{\mathcal{K}\sigma})}_{\stackrel{\text{def}}{=} \overline{F}_{\mathcal{K},\sigma}(\mathbf{u},p)}$$

CELL-CENTERED VELOCITY AND EDGE-CENTERED PRESSURE

$$\mathbf{u}^\mathcal{T} = (\mathbf{u}_\mathcal{K})_{\mathcal{K}\in\mathcal{T}} \in (\mathbb{R}^2)^\mathcal{T} \quad \text{and} \quad p^\mathcal{T} = (p_\sigma)_\sigma \in \mathbb{R}^\mathcal{E}.$$

NAIVE IDEA FOR THE NUMERICAL FLUX

$$F_{\mathcal{K},\sigma}(\mathbf{u}^\mathcal{T}, p^\mathcal{T}) \stackrel{\text{def}}{=} -|\sigma| \frac{\mathbf{u}_\mathcal{L} - \mathbf{u}_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}} + p_\sigma |\sigma| \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.$$

$$F_{\mathcal{K},\sigma}(\mathbf{u}^{\mathcal{T}}, p^{\mathcal{T}}) \stackrel{\text{def}}{=} -|\sigma| \frac{\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}} + p_\sigma |\sigma| \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.$$

- This naturally leads to the following *discrete pressure gradient operator*

$$(\nabla^{\mathcal{T}} p^{\mathcal{T}})_{\mathcal{K}} = \frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} p_\sigma |\sigma| \boldsymbol{\nu}_{\mathcal{K}\sigma}.$$

- To ensure stability/well-posedness/convergence it is required that the *discrete divergence operator* is the adjoint of the *gradient operator*.

$$(\text{div}^{\mathcal{T}} \mathbf{u}^{\mathcal{T}})_\sigma = |\sigma| (\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.$$

TWO MAIN ISSUES:

- This operator is not consistent with the divergence operator : it is an **incomplete** approximation.
- **Numerical locking** : with this definition $\mathbf{u}^{\mathcal{T}} = 0$ is the unique vector field which is divergence-free and vanishes on the boundary.

$\rightsquigarrow$ there are too many pressure dofs compared to velocity dofs.

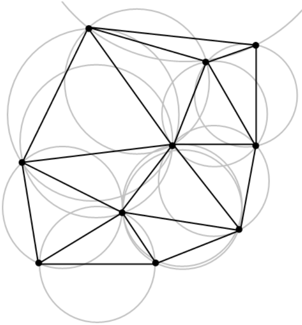
IT IS NEEDED TO DEAL WITH SLIGHTLY MORE COMPLEX METHODS

- MAC scheme : Cartesian cells
- DDFV : generalisation of MAC to more general grids
- others (SUSHI, Mimetic FD, ...)

- Cartesian meshes : Control volumes are rectangular parallelepipeds thus choosing $x_\mathcal{K}$ as the mass center is OK

- Cartesian meshes :
- Conforming triangular meshes :
  We take $x_\mathcal{K}$ =circumcenter ; **BUT :**
  - It is not guaranteed that $x_\mathcal{K} \in \mathcal{K}$ (even $x_\mathcal{K} \in \Omega$ is not sure).
  - We can have $x_\mathcal{K} = x_\mathcal{L}$ for $\mathcal{K} \neq \mathcal{L} \Rightarrow d_{\mathcal{K}\mathcal{L}} = 0$!
  - However, the TPFA scheme still works for the Laplace equation if

$$(x_\mathcal{L} - x_\mathcal{K}) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} > 0 \quad \Leftrightarrow \text{Delaunay condition}$$



Other TPFA drawbacks

- Cartesian meshes :
- Conforming triangular meshes :
- For a non conforming triangle mesh: orthogonality condition is impossible to fulfill.
- For a non Cartesian quadrangle mesh: orthogonality condition is impossible to fulfill in general.

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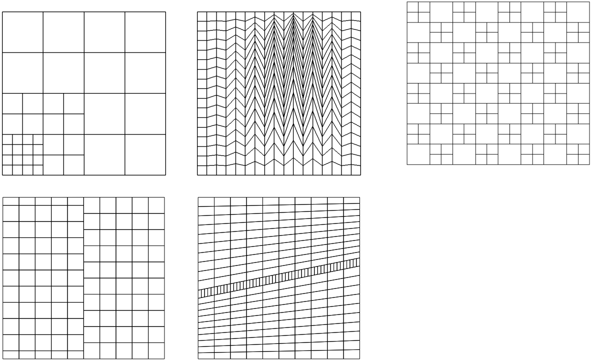
Some academic meshes that we would like to deal with



General meshes are allowed

- Possibly non conforming meshes
- Without the orthogonality condition

For scalar elliptic problems

(Hermeline '00) (Domelevo-Omnes '05) (Andreianov-Boyer-Hubert '07)

For the Stokes problem

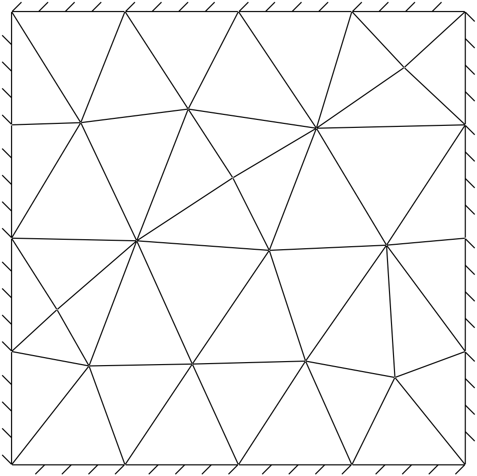
(Delcourte '07) (Krell '10) (Krell-Manzini, '12)

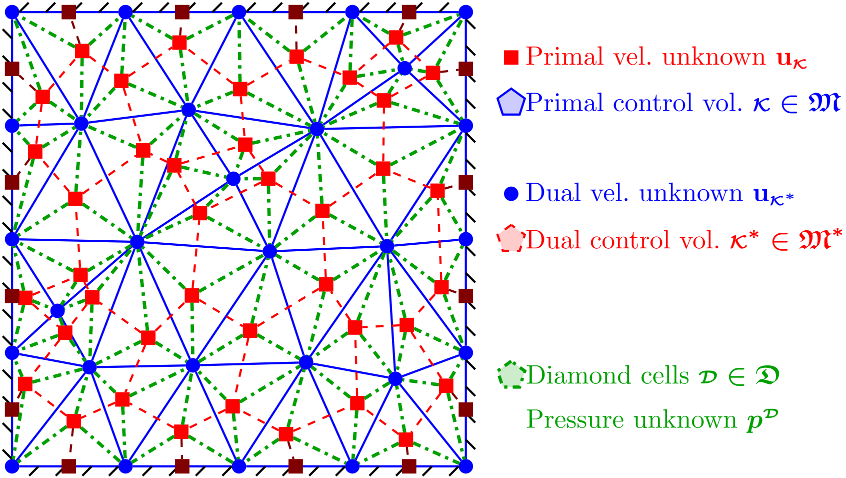
DDFV for other models

- Nonlinear elliptic problems.
- Maxwell equations.
- Drift-diffusion in semi-conductors.
- Elasticity.

Basic ideas (in the case of the Stokes problem)

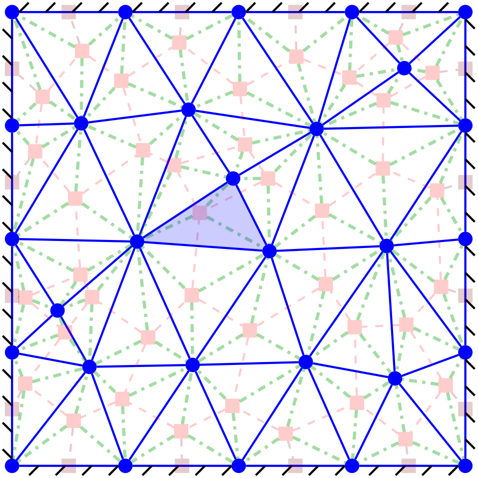
- Still consider a single pressure unknown at each edge.
- Consider velocity unknowns at the center of each control volume but also on **vertices**.
- Add new discrete balance equations associated with each vertex.
- For a Poisson equation it is more *expensive* than TPFA # unknowns ($\approx \times 2$) but much more robust and efficient.





APP. SOLUTION : $\mathbf{u}^{\mathcal{T}} = \left((\mathbf{u}_\kappa)_\kappa, (\mathbf{u}_{\kappa^*})_{\kappa^*} \right) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} = (\mathbb{R}^2)^{\mathfrak{M}} \times (\mathbb{R}^2)^{\mathfrak{M}^*}$,

$p^{\mathfrak{D}} = \left(p^{\mathcal{D}}\right)_{\mathcal{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$



- Primal vel. unknown $\mathbf{u}_{\kappa}$
- Primal control vol. $\kappa \in \mathfrak{M}$
- Dual vel. unknown $\mathbf{u}_{\kappa^*}$
- Dual control vol. $\kappa^* \in \mathfrak{M}^*$
- Diamond cells $\mathcal{D} \in \mathfrak{D}$

  Pressure unknown $p^{\mathcal{D}}$

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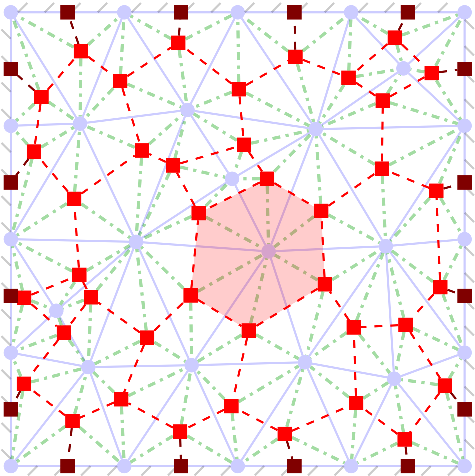

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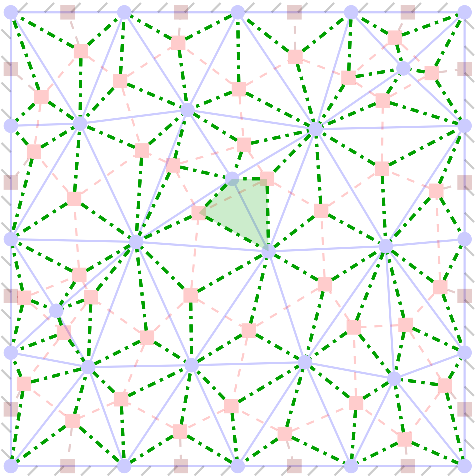

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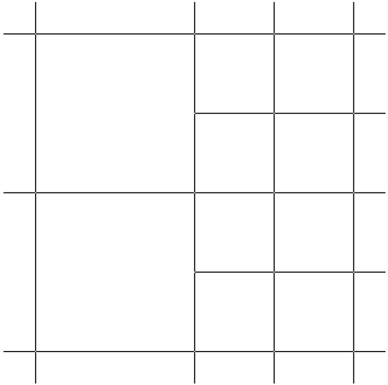

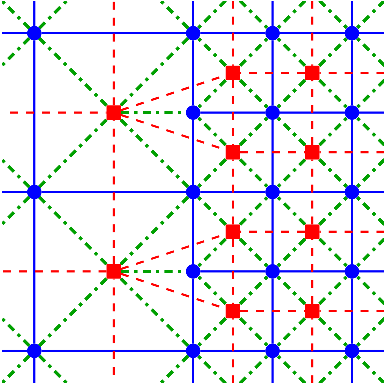

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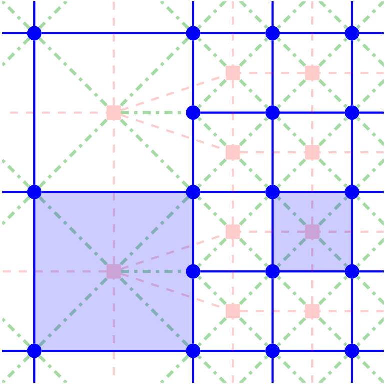

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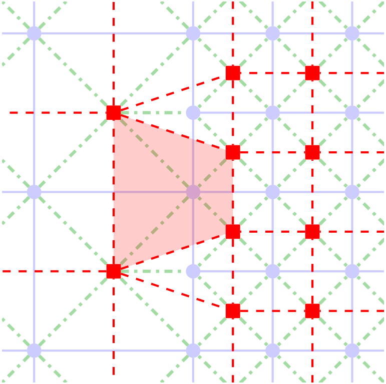

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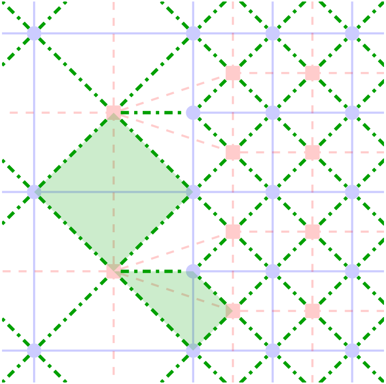

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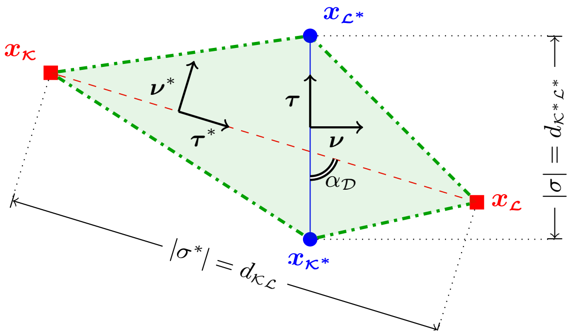
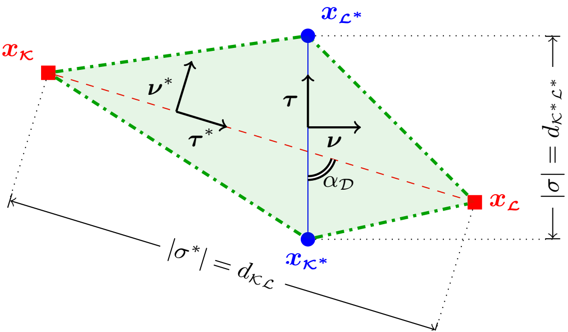


DISCRETE GRADIENT

$$\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\tau}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left[\frac{\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}}{|\sigma^*|} \otimes \boldsymbol{\nu} + \frac{\mathbf{u}_{\mathcal{L}^*} - \mathbf{u}_{\mathcal{K}^*}}{|\sigma|} \otimes \boldsymbol{\nu}^* \right].$$

COMES FROM
$$\begin{cases} \nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\tau}}.(x_{\mathcal{L}} - x_{\mathcal{K}}) = \mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}, \\ \nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\tau}}.(x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \mathbf{u}_{\mathcal{L}^*} - \mathbf{u}_{\mathcal{K}^*}. \end{cases}$$



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DISCRETE VELOCITY DIVERGENCE

$$\mathrm{div}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\tau}} = \mathrm{Tr}(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\tau}}) = \frac{1}{\sin \alpha_{\mathcal{D}}} \left[\frac{(\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}) \cdot \boldsymbol{\nu}}{|\sigma^*|} + \frac{(\mathbf{u}_{\mathcal{L}^*} - \mathbf{u}_{\mathcal{K}^*}) \cdot \boldsymbol{\nu}^*}{|\sigma|} \right].$$

We have just defined

$$\nabla^{\mathcal{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$$
$$\mathrm{div}^{\mathcal{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \mathbb{R}^{\mathfrak{D}}.$$

We need the adjoint of these operators

Discrete tensor divergence

$$\mathbf{div}^{\mathcal{T}}(\xi^{\mathfrak{D}}) = \begin{cases} \mathbf{div}^{\mathcal{K}}\xi^{\mathfrak{D}} = \dfrac{1}{|\mathcal{K}|}\displaystyle\sum_{\sigma\subset\partial\mathcal{K}} |\sigma|\xi^{\mathcal{D}}.\boldsymbol{\nu}, & \forall\ \mathcal{K}\in\mathfrak{M}, \\ \mathbf{div}^{\mathcal{K}^*}\xi^{\mathfrak{D}} = \dfrac{1}{|\mathcal{K}^*|}\displaystyle\sum_{\sigma^*\subset\partial\mathcal{K}^*} |\sigma^*|\xi^{\mathcal{D}}.\boldsymbol{\nu}^*, & \forall\ \mathcal{K}^*\in\mathfrak{M}^*. \end{cases}$$

Discrete pressure gradient

$$\boldsymbol{\nabla}^{\mathcal{T}} p^{\mathfrak{D}} = \mathbf{div}^{\mathcal{T}}(p^{\mathfrak{D}}\mathrm{Id})$$

$$\mathbf{div}^{\mathcal{T}} : \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}} \longrightarrow \left(\mathbb{R}^2\right)^{\mathcal{T}},$$
$$\boldsymbol{\nabla}^{\mathcal{T}} : \mathbb{R}^{\mathfrak{D}} \longrightarrow \left(\mathbb{R}^2\right)^{\mathcal{T}}.$$

DDFV scheme

Find $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that

$$\begin{cases} -\mathbf{div}^{\mathcal{T}}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + \boldsymbol{\nabla}^{\mathcal{T}} p^{\mathfrak{D}} = \mathbf{f}^{\mathcal{T}}, \\ \mathrm{div}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} = 0, \\ m(p^{\mathfrak{D}}) = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| p^{\mathcal{D}} = 0. \end{cases}$$

where $\mathbb{E}_0^{\mathcal{T}} = \left\{ \mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \text{ s. t. } \forall \kappa \in \partial\mathfrak{M},\ \mathbf{u}_{\kappa} = 0,\ \forall \kappa^* \in \partial\mathfrak{M}^*,\ \mathbf{u}_{\kappa^*} = 0 \right\}$.

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where $\mathbb{E}_0^{\mathcal{T}} = \left\{ \mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \text{ s. t. } \forall \mathcal{K} \in \partial\mathfrak{M},\ \mathbf{u}_{\mathcal{K}} = 0,\ \forall \mathcal{K}^* \in \partial\mathfrak{M}^*,\ \mathbf{u}_{\mathcal{K}^*} = 0 \right\}$.

Primal cell equations for orthogonal meshes

$$\sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{\mathcal{K}}} \left(-|\sigma| \frac{\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}} + p^{\mathcal{D}} |\sigma| \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right) = |\mathcal{K}| \mathbf{f}_{\mathcal{K}}.$$

- This is a two-point flux approximation !
- A similar equation holds on dual cells.

The divergence-free equation for orthogonal meshes

$$\frac{(\mathbf{u}_{\mathcal{L}} - \mathbf{u}_{\mathcal{K}}) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}}{|\sigma^*|} + \frac{(\mathbf{u}_{\mathcal{L}^*} - \mathbf{u}_{\mathcal{K}^*}) \cdot \boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}}{|\sigma|} = 0, \quad \text{for each diamond cell } \mathcal{D}.$$

- **(S. Krell, PhD '10)** $\rightsquigarrow$ No Inf-Sup stability proof
  $\Rightarrow$ However she gives convergence proofs and error estimates by adding a stabilisation term in the mass conservation equation.

- Numerical experiments $\rightsquigarrow$ No need of stabilisation ?
  $\Rightarrow$ We are going to study the behavior of the Inf-Sup (LLB) constant.

Two different aspects

1. **Numerical study** : For a given mesh, we want to compute the discrete Inf-Sup constant and observe its behavior when refining the mesh.
   $\Rightarrow$ Eigenvalue problem.
2. **Theoretical study** : Can we prove theorems to confirm numerical evidences ?

DISCRETE INF-SUP CONSTANT

$$\beta_{\mathcal{T}} = \inf_{p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{(\mathrm{div}^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}})_{\mathfrak{D}}}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \|p^{\mathfrak{D}} - m(p^{\mathfrak{D}})\|_{\mathfrak{D},2}} \right)$$

THE QUESTIONS TO BE INVESTIGATED

1. **Well-posedness of the scheme** : For a given mesh do we have $\beta_{\mathcal{T}} > 0$?
2. **Stability** : For a given family of meshes, do we have

$$\liminf_{\mathrm{size}(\mathcal{T}) \to 0} \beta_{\mathcal{T}} > 0 \,?$$

Lemma

The DDFV scheme can be written as follows

$$\begin{cases} \begin{pmatrix} R_{\mathcal{T}} & {}^t B_{\mathcal{T},\mathfrak{D}} \\ B_{\mathcal{T},\mathfrak{D}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^{\mathcal{T}} \\ p^{\mathfrak{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\mathcal{T}} \\ 0 \end{pmatrix}, \\ \\ \langle M_{\mathfrak{D}} p^{\mathfrak{D}}, {}^t\mathbf{1} \rangle = 0. \end{cases}$$

- $R_{\mathcal{T}}$ *is the DDFV stiffness matrix.*
- $B_{\mathcal{T},\mathfrak{D}}$ *is the discrete divergence matrix.*
- $M_{\mathfrak{D}}$ *is the pressure mass matrix.*

New expression of the Inf-Sup constant

$$\beta_{\mathcal{T}} = \inf_{\substack{p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \\ m(p^{\mathfrak{D}})=0}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{\langle B_{\mathcal{T},\mathfrak{D}} \mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}} \rangle}{\langle R_{\mathcal{T}} \mathbf{v}^{\mathcal{T}}, \mathbf{v}^{\mathcal{T}} \rangle^{\frac{1}{2}} \langle M_{\mathfrak{D}} p^{\mathfrak{D}}, p^{\mathfrak{D}} \rangle^{\frac{1}{2}}} \right).$$

Lemma

The DDFV scheme can be written as follows

$$\begin{cases} \begin{pmatrix} R_{\mathcal{T}} & {}^t B_{\mathcal{T},\mathfrak{D}} \\ B_{\mathcal{T},\mathfrak{D}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^{\mathcal{T}} \\ p^{\mathfrak{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\mathcal{T}} \\ 0 \end{pmatrix}, \\ \langle M_{\mathfrak{D}} p^{\mathfrak{D}}, {}^t\mathbf{1} \rangle = 0. \end{cases}$$

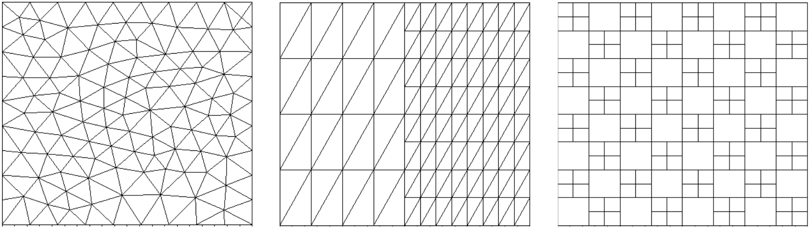
- $R_{\mathcal{T}}$ *is the DDFV stiffness matrix.*
- $B_{\mathcal{T},\mathfrak{D}}$ *is the discrete divergence matrix.*
- $M_{\mathfrak{D}}$ *is the pressure mass matrix.*

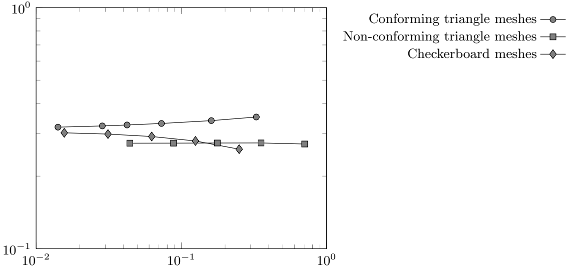
Theorem (Schur complement formulation)

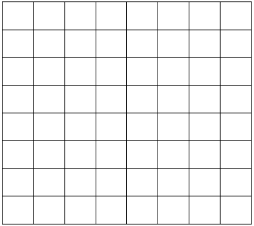
We have

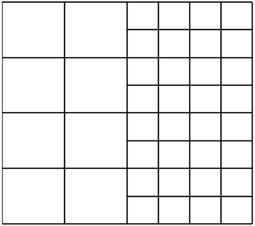
$$\beta_{\mathcal{T}}^2 = \textit{second smallest eigenvalue of } S_{\mathfrak{D}},$$

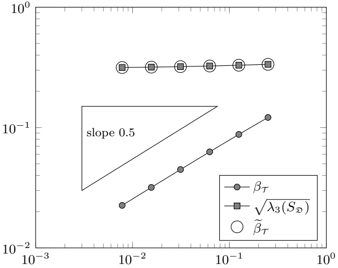
where $S_{\mathfrak{D}} = M_{\mathfrak{D}}^{-\frac{1}{2}} B_{\mathcal{T},\mathfrak{D}} R_{\mathcal{T}}^{-1} {}^t B_{\mathcal{T},\mathfrak{D}} M_{\mathfrak{D}}^{-\frac{1}{2}}$.



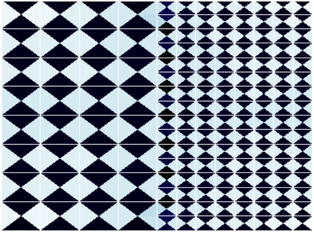

10^0
10^{-1}
10^{-2}
10^{-1}
10^0
Conforming triangle meshes
Non-conforming triangle meshes
Checkerboard meshes

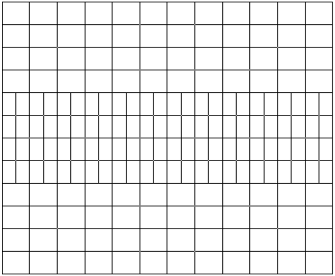


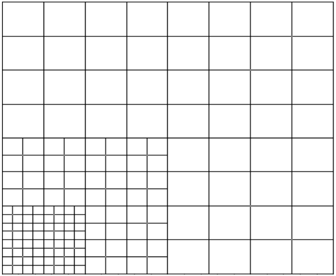


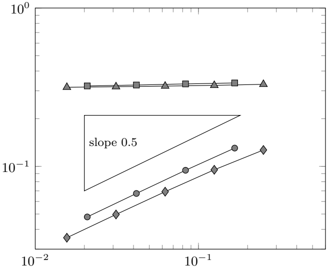


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^0
slope 0.5
$\beta_{\mathcal{T}}$
$\sqrt{\lambda_3(S_{\mathfrak{D}})}$
$\widetilde{\beta}_{\mathcal{T}}$

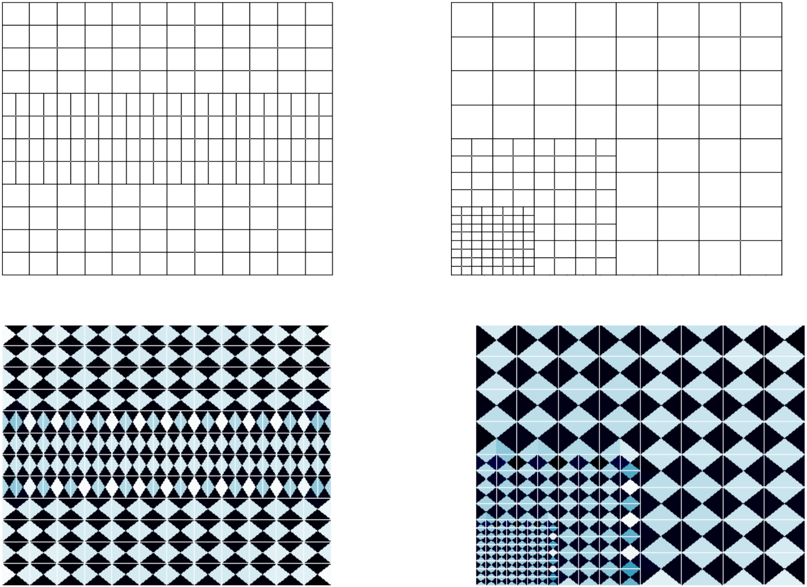


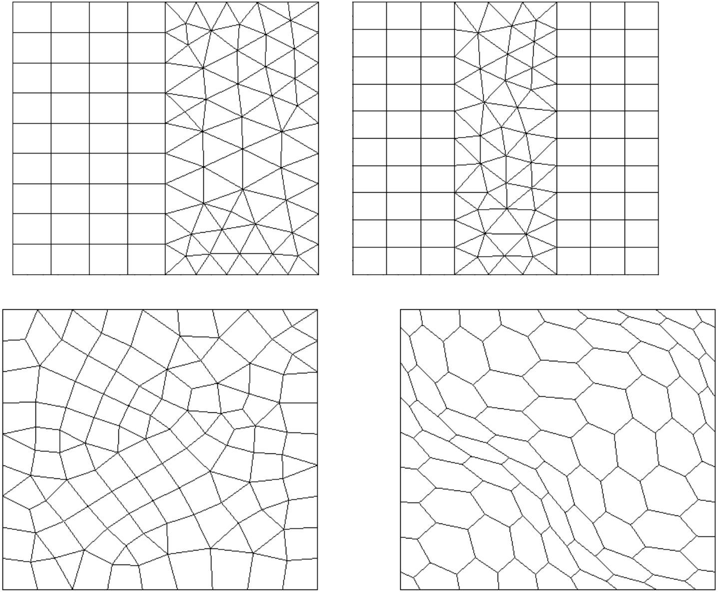


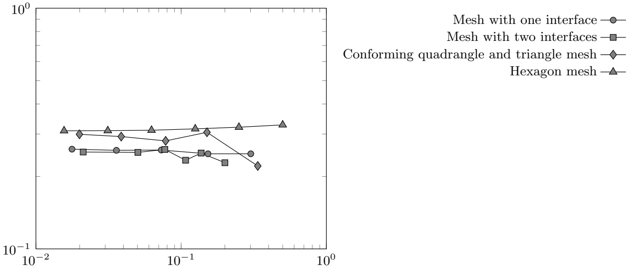




$\beta_{\mathcal{T}}$ for the mesh with two interfaces
$\sqrt{\lambda_3(S_{\mathfrak{D}})}$ for the mesh with two interfaces
$\beta_{\mathcal{T}}$ for the locally refined mesh
$\sqrt{\lambda_3(S_{\mathfrak{D}})}$ for the locally refined mesh






10^0
10^{-1}
10^{-2}
10^{-1}
10^0
Mesh with one interface
Mesh with two interfaces
Conforming quadrangle and triangle mesh
Hexagon mesh

LEMMA (S. Krell,'10)

There exists a (Fortin-like) operator $\mathbf{v} \in (H^1_0(\Omega))^2 \longrightarrow \mathbf{v}^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, *and a* $C > 0$ *such that for any* $\mathbf{v} \in (H^1_0(\Omega))^2$ *and* $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, *we have*

$$\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \leq C\|\nabla \mathbf{v}\|_{L^2},$$

$$\left|\sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}} \mathbf{v}^{\mathcal{T}} - \mathrm{div}\mathbf{v}\right) dz\right| \leq C|p^{\mathfrak{D}}|_h \|\mathbf{v}\|_{H^1}, \qquad (\star)$$

with $|p^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} ({h_{\mathcal{D}}}^2 + {h_{\mathcal{D}'}}^2)(p^{\mathcal{D}'} - p^{\mathcal{D}})^2, \qquad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$

USUAL FORTIN'S PROOF :

- **Imagine that the r.h.s. in $(\star)$ is 0.**
- Pick any $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ s.t. $\int_\Omega p^{\mathfrak{D}} = 0$ and apply Nečas' inequality

  There exists a $\mathbf{v} \in (H^1_0(\Omega))^2$ s.t. $\mathrm{div}\mathbf{v} = p^{\mathfrak{D}}$, and $\|\nabla \mathbf{v}\|_{L^2} \leq C_2\|p^{\mathfrak{D}}\|_{L^2}$.
- Apply $(\star)$

$$\left.\begin{array}{l} \displaystyle\int_\Omega |p^{\mathfrak{D}}|^2 = \int_\Omega p^{\mathfrak{D}} \mathrm{div}\mathbf{v} = \int_\Omega p^{\mathfrak{D}} \mathrm{div}^{\mathcal{D}} \mathbf{v}^{\mathcal{T}} \\ \|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \leq C\|\nabla \mathbf{v}\|_{L^2} \leq CC_2\|p^{\mathfrak{D}}\|_{\mathfrak{D},2} \end{array}\right\} \Rightarrow \|p^{\mathfrak{D}}\|_{L^2} \leq CC_2 \frac{\displaystyle\int_\Omega p^{\mathfrak{D}} \mathrm{div}^{\mathcal{D}} \mathbf{v}^{\mathcal{T}}}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2}}.$$

Lemma (S. Krell,'10)

There exists a (Fortin-like) operator $\mathbf{v} \in (H^1_0(\Omega))^2 \longrightarrow \mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \mathbb{R}^{\mathcal{T}}$, *and a* $C > 0$ *such that for any* $\mathbf{v} \in (H^1_0(\Omega))^2$ *and* $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, *we have*

$$\|\nabla^{\mathfrak{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}\|_{\mathfrak{D},2} \leq C \|\nabla \mathbf{v}\|_{L^2},$$

$$\left| \sum_{\mathcal{D} \in \mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}} - \mathrm{div} \mathbf{v} \right) dz \right| \leq C |p^{\mathfrak{D}}|_h \|\mathbf{v}\|_{H^1}, \qquad (\star)$$

with $|p^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{S}} ({h_{\mathcal{D}}}^2 + {h_{\mathcal{D}'}}^2)(p^{\mathcal{D}'} - p^{\mathcal{D}})^2, \qquad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$

Here we need do deal with the r.h.s term in $(\star)$

LEMMA (S. Krell,'10)

There exists a (Fortin-like) operator $\mathbf{v} \in (H^1_0(\Omega))^2 \longrightarrow \mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \mathbb{R}^{\mathcal{T}}$, *and a* $C > 0$ *such that for any* $\mathbf{v} \in (H^1_0(\Omega))^2$ *and* $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, *we have*

$$\|\nabla^{\mathfrak{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}\|_{\mathfrak{D},2} \leq C\|\nabla \mathbf{v}\|_{L^2},$$

$$\left| \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}} - \mathrm{div}\mathbf{v}\right) dz \right| \leq C|p^{\mathfrak{D}}|_h \|\mathbf{v}\|_{H^1}, \qquad (\star)$$

$$\textit{with } |p^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} ({h_{\mathcal{D}}}^2 + {h_{\mathcal{D}'}}^2)(p^{\mathcal{D}'} - p^{\mathcal{D}})^2, \qquad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$$

PROPOSITION

Assume that for some $\alpha_{\mathcal{T}} \geq 1$ *we can prove that*

$$|p^{\mathfrak{D}}|_h \leq \alpha_{\mathcal{T}} \|h^{\mathcal{T}} \boldsymbol{\nabla}^{\boldsymbol{\mathcal{T}}} p^{\mathfrak{D}}\|_{\mathcal{T},2}, \quad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}. \qquad (\star\star)$$

Then, the Inf-Sup inequality holds with $\beta_{\mathcal{T}} = C/\alpha_{\mathcal{T}}$.

- For $p^{\mathfrak{D}}$ such that $\|h^{\mathcal{T}} \boldsymbol{\nabla}^{\boldsymbol{\mathcal{T}}} p^{\mathfrak{D}}\|_{\mathcal{T},2} \leq M \|p^{\mathfrak{D}}\|_{\mathfrak{D},2} \Rightarrow$ Use Fortin's like proof
- For $p^{\mathfrak{D}}$ such that $\|h^{\mathcal{T}} \boldsymbol{\nabla}^{\boldsymbol{\mathcal{T}}} p^{\mathfrak{D}}\|_{\mathcal{T},2} \geq M \|p^{\mathfrak{D}}\|_{\mathfrak{D},2}$
  $\Rightarrow$ Just check that $\mathbf{v}^{\boldsymbol{\mathcal{T}}} = h^{\mathcal{T}} \boldsymbol{\nabla}^{\boldsymbol{\mathcal{T}}} p^{\mathfrak{D}}$ is OK

Our problem

We want to prove that pressure differences between neighboring diamonds are controlled by the DDFV pressure gradient.

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} h^2(p^{\mathcal{D}'}-p^{\mathcal{D}})^2 \leq C\left(\sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}| h_{\mathcal{K}}^2 |\nabla^{\mathcal{K}} p^{\mathfrak{D}}|^2 + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\mathcal{K}^*| h_{\mathcal{K}^*}^2 |\nabla^{\mathcal{K}^*} p^{\mathfrak{D}}|^2\right).$$

Notations

Let be $\mathcal{K}$ a primal cell, $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ three diamond cells of $\mathcal{K}$.

- We say that $\boxed{p^{\mathcal{D}} \xrightarrow{\mathcal{K}} p^{\mathcal{D}'}}$ if there is a (uniform) C such that

$$|p^{\mathcal{D}} - p^{\mathcal{D}'}| \leq C h_{\mathcal{K}} \left|\nabla^{\mathcal{K}} p^{\mathfrak{D}}\right|.$$

- We say that $\boxed{\begin{cases} p^{\mathcal{D}'} \\ p^{\mathcal{D}''} \end{cases} \xrightarrow{\mathcal{K}} p^{\mathcal{D}}}$ if we have $p^{\mathcal{D}} \xrightarrow{\mathcal{K}} p^{\mathcal{D}'}$ **and** $p^{\mathcal{D}} \xrightarrow{\mathcal{K}} p^{\mathcal{D}''}$.

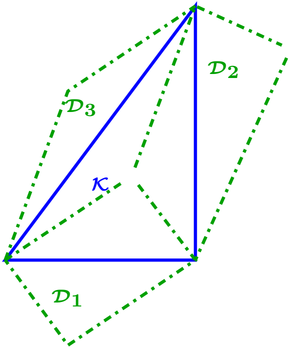
- We say that $\boxed{p^{\mathcal{D}} \xrightarrow{\mathcal{K}} \begin{cases} p^{\mathcal{D}'} \\ p^{\mathcal{D}''} \end{cases}}$ if there is a C and a $\theta \in [0,1]$ such that

$$|\theta(p^{\mathcal{D}} - p^{\mathcal{D}'}) + (1-\theta)(p^{\mathcal{D}} - p^{\mathcal{D}''})| \leq C h_{\mathcal{K}} \left|\nabla^{\mathcal{K}} p^{\mathfrak{D}}\right|.$$

- Similar notations for dual cells $\mathcal{K}^* \in \mathfrak{M}^*$.

Theorem

Any regular conforming triangle mesh is Inf-Sup stable.



DDFV Gradient definition

$$|\mathcal{K}|\nabla^{\mathcal{K}}p^{\mathfrak{D}} = \sum_{i=1}^{3} m_{\sigma_i} p^{\mathcal{D}_i} \vec{\mathbf{n}}_{\sigma_i\mathcal{K}},$$

"Geometric" formula

$$\sum_{i=1}^{3} m_{\sigma_i} \vec{\mathbf{n}}_{\sigma_i\mathcal{K}} = 0.$$

Conclusion

$$|\mathcal{K}|\nabla^{\mathcal{K}}p^{\mathfrak{D}} = m_{\sigma_1}(p^{\mathcal{D}_1} - p^{\mathcal{D}_3})\vec{\mathbf{n}}_{\sigma_1\mathcal{K}} + m_{\sigma_2}(p^{\mathcal{D}_2} - p^{\mathcal{D}_3})\vec{\mathbf{n}}_{\sigma_2\mathcal{K}}.$$

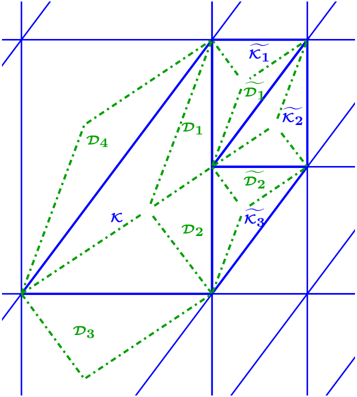
$$\Longrightarrow |p^{\mathcal{D}_1} - p^{\mathcal{D}_3}| = \frac{m_{\sigma_2}}{2}\left|\nabla^{\mathcal{K}}p^{\mathfrak{D}} \wedge \vec{\mathbf{n}}_{\sigma_2\mathcal{K}}\right| \leq C h_{\mathcal{K}} \left|\nabla^{\mathcal{K}}p^{\mathfrak{D}}\right|,$$

that is

$$p^{\mathcal{D}_1} \overset{\mathcal{K}}{\longrightarrow} p^{\mathcal{D}_3}.$$

Theorem

A “bidomain” non conforming triangle mesh family is Inf-Sup stable.



Observation

$$p^{\mathcal{D}_1} \xrightarrow{\mathcal{K}} p^{\mathcal{D}_2}, \text{does not hold.}$$

Using neighboring cells

$$p^{\mathcal{D}_1} \xrightarrow{\widetilde{\mathcal{K}_1}} p^{\widetilde{\mathcal{D}_1}} \xrightarrow{\widetilde{\mathcal{K}_2}} p^{\widetilde{\mathcal{D}_2}} \xrightarrow{\widetilde{\mathcal{K}_3}} p^{\mathcal{D}_2},$$

The same analysis as before

$$p^{\mathcal{D}_3} \xrightarrow{\mathcal{K}} p^{\mathcal{D}_4}$$

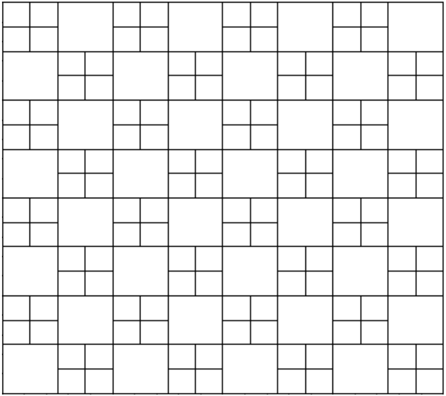
$$\frac{p^{\mathcal{D}_1} + p^{\mathcal{D}_2}}{2} \xrightarrow{\mathcal{K}} p^{\mathcal{D}_3}$$

$$\frac{p^{\mathcal{D}_1} + p^{\mathcal{D}_2}}{2} \xrightarrow{\mathcal{K}} p^{\mathcal{D}_4}$$

$\rightsquigarrow$ We can control all the pressure differences $p^{\mathcal{D}_i} - p^{\mathcal{D}_j}$.

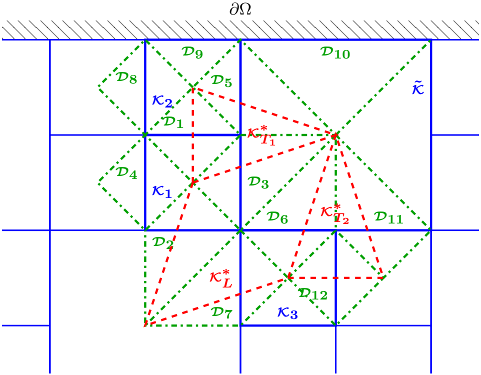
Theorem

The checkerboard-like family of meshes is Inf-Sup stable.



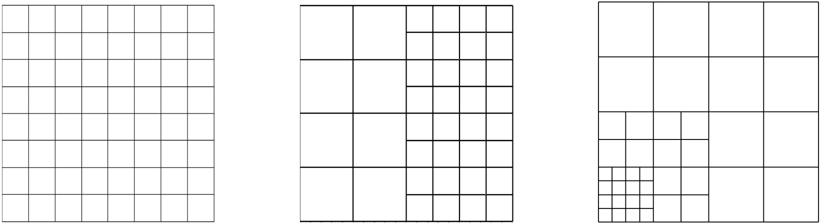
THEOREM

The checkerboard-like family of meshes is Inf-Sup stable.



$$p^{\mathcal{D}_{10}} \xrightarrow{\widetilde{\mathcal{K}}} \begin{cases} p^{\mathcal{D}_6} \\ p^{\mathcal{D}_{11}} \end{cases} \xrightarrow{\mathcal{K}^*_{T_2}} p^{\mathcal{D}_{12}} \xrightarrow{\mathcal{K}_3} p^{\mathcal{D}_7} \xrightarrow{\mathcal{K}^*_L} p^{\mathcal{D}_3} \xrightarrow{\mathcal{K}^*_{T_1}} p^{\mathcal{D}_5},$$

Cartesian meshes



Definition (Checkerboard mode)

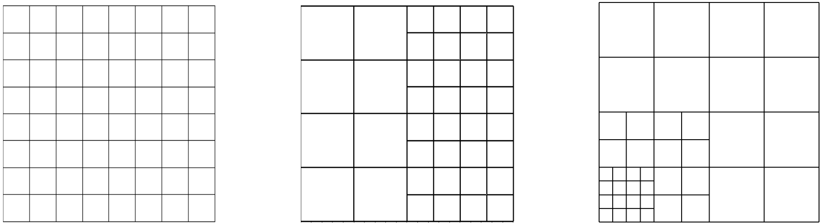
Let us define $\psi^{\mathfrak{D}}$ by $\psi^{\mathcal{D}} = \begin{cases} +1, & \text{for } \mathcal{D} \in \mathfrak{D}^v, \\ -1, & \text{for } \mathcal{D} \in \mathfrak{D}^h. \end{cases}$

Theorem (Uniform Cartesian meshes)

We have

$$\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, \psi^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2}} = 0.$$

Thus $\beta_{\mathcal{T}} = 0$ $\rightsquigarrow$ **The scheme is ill-posed.**



Definition (Checkerboard mode)

Let us define $\psi^{\mathfrak{D}}$ by $\psi^{\mathcal{D}} = \begin{cases} +1, & \text{for } \mathcal{D} \in \mathfrak{D}^v, \\ -1, & \text{for } \mathcal{D} \in \mathfrak{D}^h. \end{cases}$

Theorem (Nonconforming Cartesian meshes)

We have

$$\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, \psi^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2}} \approx \; size(\mathcal{T})^{\frac{1}{2}}.$$

Thus $\beta_{\mathcal{T}} \leq C \; size(\mathcal{T})^{\frac{1}{2}}$ $\rightsquigarrow$ **The scheme is well-posed but unstable.**

Looking at the pressure fields orthogonal to $\psi^{\mathfrak{D}}$

$$\widetilde{\beta}_{\mathcal{T}} = \inf_{\substack{p^{\mathfrak{D}} \in \{\psi^{\mathfrak{D}}\}^{\perp} \\ m(p^{\mathfrak{D}})=0}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \|p^{\mathfrak{D}}\|_{\mathfrak{D},2}} \right)$$

Theorem

For any Cartesian (uniform or nonconforming) meshes family, we have

$$\liminf_{size(\mathcal{T}) \to 0} \widetilde{\beta}_{\mathcal{T}} > 0.$$

Sketch of proof

- Split each pressure field $p^{\mathfrak{D}}$ into two parts

$$p^{\mathfrak{D}^v} = \sum_{\mathcal{D} \in \mathfrak{D}^v} p^{\mathcal{D}} 1_{\mathcal{D}}, \quad \text{and} \quad p^{\mathfrak{D}^h} = \sum_{\mathcal{D} \in \mathfrak{D}^h} p^{\mathcal{D}} 1_{\mathcal{D}},$$

- Both parts have a zero average (since $p^{\mathfrak{D}} \perp \psi^{\mathfrak{D}}$)

$$\Rightarrow \exists \mathbf{v}^v, \mathbf{v}^h \in (H_0^1(\Omega))^2, \text{ s.t. } \operatorname{div} \mathbf{v}^v = p^{\mathfrak{D}^v}, \text{ and } \operatorname{div} \mathbf{v}^h = p^{\mathfrak{D}^h}.$$

- Project both velocity fields $\mathbf{v}^v$ and $\mathbf{v}^h$ onto the mesh to get a suitable $\mathbf{v}^{\mathcal{T}}$ that fulfills a suitable Fortin-like estimate.

LOOKING AT THE PRESSURE FIELDS ORTHOGONAL TO $\psi^{\mathfrak{D}}$

$$\widetilde{\beta}_{\mathcal{T}} = \inf_{\substack{p^{\mathfrak{D}} \in \{\psi^{\mathfrak{D}}\}^{\perp} \\ m(p^{\mathfrak{D}})=0}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \|p^{\mathfrak{D}}\|_{\mathfrak{D},2}} \right)$$

THEOREM

For any Cartesian (uniform or nonconforming) meshes family, we have

$$\liminf_{size(\mathcal{T}) \to 0} \widetilde{\beta}_{\mathcal{T}} > 0.$$

CONCLUSIONS

- There is one single unstable pressure mode.
- This pressure mode looks like the checkerboard mode $\psi^{\mathfrak{D}}$.
- For smooth solutions, this unstable mode does not contribute to the error.

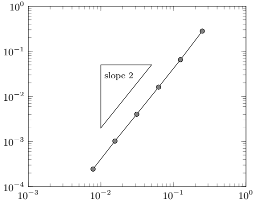
Looking at the pressure fields orthogonal to $\psi^{\mathfrak{D}}$

$$\widetilde{\beta}_{\mathcal{T}} = \inf_{\substack{p^{\mathfrak{D}} \in \{\psi^{\mathfrak{D}}\}^{\perp} \\ m(p^{\mathfrak{D}})=0}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \|p^{\mathfrak{D}}\|_{\mathfrak{D},2}} \right)$$

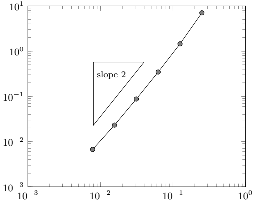
Theorem

For any Cartesian (uniform or nonconforming) meshes family, we have

$$\liminf_{size(\mathcal{T}) \to 0} \widetilde{\beta}_{\mathcal{T}} > 0.$$



L^2-Error for $\mathbf{v}$



L^2-Error for p

LOOKING AT THE PRESSURE FIELDS ORTHOGONAL TO $\psi^{\mathfrak{D}}$

$$\widetilde{\beta}_{\mathcal{T}} = \inf_{\substack{p^{\mathfrak{D}} \in \{\psi^{\mathfrak{D}}\}^{\perp} \\ m(p^{\mathfrak{D}})=0}} \left(\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, p^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2} \|p^{\mathfrak{D}}\|_{\mathfrak{D},2}} \right)$$

THEOREM

For any Cartesian (uniform or nonconforming) meshes family, we have

$$\liminf_{size(\mathcal{T}) \to 0} \widetilde{\beta}_{\mathcal{T}} > 0.$$

THEOREM (UNSTABLE MODE STRUCTURE)

Let $q^{\mathfrak{D}}$ *such that* $m(q^{\mathfrak{D}}) = 0$, $\|q^{\mathfrak{D}}\|_{\mathfrak{D},2} = 1$, $(q^{\mathfrak{D}}, \psi^{\mathfrak{D}}) > 0$ *and*

$$\sup_{\mathbf{v}^{\mathcal{T}} \in \mathbb{E}_0^{\mathcal{T}}} \frac{b_{\mathcal{T}}(\mathbf{v}^{\mathcal{T}}, q^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathcal{T}}\|_{\mathfrak{D},2}} = \beta_{\mathcal{T}}.$$

Then we have,

$$\|q^{\mathfrak{D}} - \psi^{\mathfrak{D}}\|_{\mathfrak{D},2} \leq C \ size(\mathcal{T})^{\frac{1}{2}}, \quad and \quad \beta_{\mathcal{T}} \sim C \ size(\mathcal{T})^{\frac{1}{2}}.$$

Conclusions

Main properties of DDFV

- Easy implementation.
- Suitable for very general meshes.
- Very robust : even though it can be Inf-Sup stable in some particular cases, the number of unstable modes is low (1 single mode in practice).
- Second order on locally refined Cartesian grids.

Variants / Extensions

- Variable viscosity Stokes problem **(Krell, PhD '10)**
- 3D case **(Krell-Manzini, '12)**
- Incompressible Navier-Stokes equations **(Krell, PhD '10)** **(Goudon-Krell, '14)**
- Coupling AMR/DDFV with free surface evolution through a level-set approach. **(Galusinski-Golay-Lakhlili,'14)**

Open problems