On the relative relative energy method for the compressible Navier-Stokes system

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Part 1 : Compressible Navier-Stokes equations on thin domains
based on joint work with A. Novotný
Part 2 : Towards numerical schemes and error estimates
based on work in progress with T. Gallouet, R. Herbin, A. Novotný

Compressible Navier-Stokes equations

We consider in $[0,T) \times \Omega$, $\Omega \subset R^3$ (a bounded Lipschitz domain) the following system of equations

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \tag{2}$$

Boundary conditions

$$\mathbf{u}\Big|_{(0,T)\times\partial\Omega} = 0 \tag{3}$$

Initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad \varrho\mathbf{u}(0,x) = \varrho_0\mathbf{u}_0(x). \tag{4}$$

Viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \quad (5)$$

Pressure 1 : monotonicity and regularity

$$p \in C[0, \infty) \cap C^2(0, \infty), \quad p'(\varrho) > 0 \quad (6)$$

Pressure 2 : growth at infinity

$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = p_\infty > 0, \quad \gamma > \frac{3}{2} \quad (7)$$

Helmholtz function H

$$\varrho H'(\varrho) - H(\varrho) = p(\varrho), \quad H(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s$$

Relative (potential) energy function E

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$E(\varrho, r) \geq 0, \quad E(\varrho, r) = 0 \Leftrightarrow \varrho = r$$

Weak solutions

Functional spaces

$\varrho(t,x) \geq 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$,
$\varrho \mathbf{u} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3))$, $\varrho \mathbf{u}^2 \in L^\infty(0,T; L^1(\Omega))$,
$\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3))$.

Continuity equation

$\varrho \in C_{\text{weak}}([0,T]; L^\gamma(\Omega))$ and equation (1) is replaced by the family of integral identities

$$\int_\Omega \varrho\varphi \, \mathrm{d}x \Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{8}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^1([0,T] \times \overline{\Omega})$;

Momentum equation

$\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3))$ and momentum equation (2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \,{\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi\Big)\,{\rm d}x{\rm d}t \tag{9}$$

$$+\int_0^\tau\int_\Omega \Big(p(\varrho){\rm div}_x\varphi - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\Big)\,{\rm d}x\,{\rm d}t$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^1_c([0,T]\times\Omega;R^3)$;

Energy inequality

$$\int_\Omega \Big(\frac{1}{2}\varrho\mathbf{u}^2 + E(\varrho,\overline{\varrho})\Big)\,{\rm d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,{\rm d}x{\rm d}t \le 0, \tag{10}$$

for a.a. $\tau \in (0,T)$, where $\overline{\varrho} > 0$.

Existence of weak solution

Finite energy initial data

$$0 \neq \varrho_0 \geq 0, \ \int_\Omega \frac{1}{2}\varrho_0 \mathbf{u}_0^2 + E(\varrho_0 \,|\, \overline{\varrho}) \,\mathrm{d}x < \infty. \tag{11}$$

Weak solutions : Lions,98 ($\gamma \geq \frac{9}{5}$), Feireisl, Petzeltova, N., 02 ($\gamma > \frac{3}{2}$)

Under assumptions on the initial data (11) and pressure (6), (7) with $\gamma > 3/2$, the compressible Navier-Stokes system (1–5) admits at least one weak solution.

Dissipative solutions

Relative entropy

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right) \,\mathrm{d}x$$

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x\Big(\mathbf{u} - \mathbf{U}\Big) \,\mathrm{d}x \,\mathrm{d}t$$

$$\leq \mathcal{E}\left(\varrho_0, \mathbf{u}_0 \,\middle|\, r(0), \mathbf{U}(0)\right) + \int_0^\tau \mathcal{R}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big) \mathrm{d}t$$

where the remainder $\mathcal{R}$ is given by the r.h.s. of formula (13) and the test functions are the same as in formula (13).

Relative entropy (relative energy) inequality

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right) \ \mathrm{d}x \Big|_0^\tau \qquad (12)$$

$$+ \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x\Big(\mathbf{u} - \mathbf{U}\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x\Big(\mathbf{U} - \mathbf{u}\Big) \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left(\frac{r - \varrho}{r} \partial_t p(r) - \frac{\varrho}{r} u \cdot \nabla_x p(r) \right) \ \mathrm{d}x \ \mathrm{d}t$$

for all

$$r \in C^1_c([0,T] \times \overline{\Omega}), \ r > 0, \ \mathbf{U} \in C^1_c([0,T] \times \Omega).$$

Existence of dissipative solutions

Dissipative solutions : Feireisl, Sun, N., 2011

Under assumptions on initial data (11) and pressure (6), (7) with $\gamma > 3/2$, the compressible Navier-Stokes system (1–5) admits at least one dissipative solution.

Weak solutions are dissipative : Feireisl, Jin, N., 2012

Under assumptions on initial data(11) and pressure (6), any weak solution of the compressible Navier-Stokes system (1–5) is a dissipative one.

Relative entropy with (r, U) strong solution of CNSE

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right) \ \mathrm{d}x \Big|_0^\tau \qquad (13) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x \Big(\mathbf{u} - \mathbf{U}\Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega (\varrho - r) \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}\Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U}\Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \frac{r - \varrho}{r} (\mathbf{u} - U) \cdot \nabla_x p(r) \ \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Relative entropy and stability

Weak strong uniqueness, stability Feireisl, Jin, N. 2012

Let the pressure be $C^2(0,\infty)$ function satisfying (6), (7) with $\gamma \geq \frac{3}{2}$. Let $(\varrho, \mathbf{u})$ be a weak solution to the compressible Navier-Stokes equations (1-5) emanating from the initial data $(\varrho_0, \mathbf{u}_0)$, and let $(r, \mathbf{U})$ be a strong solution of the same system emanating from the initial data $(r_0, \mathbf{U}_0)$. Then there exists $c = c(\Omega, T, \|r^{-1}\|_{0,\infty}, \|r\|_{1,\infty}, \|\mathbf{U}\|_{1,\infty})$ such that

$$\mathcal{E}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right) \leq c\mathcal{E}\left(\varrho_0, \mathbf{u}_0 \;\middle|\; r_0, \mathbf{U}_0\right).$$

Some bibliographic remarks (non exhausting)

Relative entropy method was introduced to the fluid mechanics by Dafermos (90's) in the context of conservation laws. It was broadly used for the Boltzman equation (Golse, Saint-Raymond, Ukai, ...). Inequalities of this type were employed ad-hoc with specific test functions in the case of compressible Navier-Stokes equations for the investigation of low Mach-high Reynolds number limits (Masmoudi, Jiang, Wang, ...). Weak strong uniqueness is a challenging problem mentioned in Lions (98), tempted by Desjardin, 2002 Germain, 2008, who obtained conditional results. The notion of dissipative solutions has been introduced by Lions (98) for the incompressible euler equations, and weak-strong uniqueness in this case has been proved. Theorems on weak solutions, dissipative solutions, relative entropy inequality, weak-strong uniqueness, ... can be obtained also for the complete Navier-Stokes-Fourier system (describing heat conducting flows) in the conservation of energy formulated in terms of the balance of entropy, Feiresl, N., 2009,2012. Can be generalized to unbounded domains and other boundary conditions Jesslé, Jin. Weak-strong uniqueness theorem can be viewed as an "compressible" counterpart of celebrated Prodi-Serrin conditions for incompressible Navier-Stokes equations.

Compressible Navier-Stokes equations on thin domains

We consider the barotropic Navier-Stokes system describing the motion of a compressible viscous fluid confined to a straight layer $\Omega_\varepsilon = \omega \times (0,\varepsilon)$, where ω is a particular 2-D domain (a periodic cell, bounded domain or the whole $2-D$ space).

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T)\times\Omega_\varepsilon, \tag{14}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho) = \mathrm{div}\mathbb{S}(\nabla\mathbf{u}) \quad \text{in } (0,T)\times\Omega_\varepsilon. \tag{15}$$

Equations are completed with the initial conditions

$$\varrho(0,x) = \tilde\varrho_{0,\varepsilon}(x), \quad \mathbf{u}(0,x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x), \quad x\in\Omega_\varepsilon \tag{16}$$

and boundary conditions that will be specified later.

Goal : investigate the limit process $\varepsilon \to 0$, provided the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}](x)$ converge in a certain sense to $[r_0, \mathbf{v}_0](x) = [\mathbf{v}_{0,h}, 0](x_h)$.
Will the sequence $[\varrho_\varepsilon, \mathbf{u}_\varepsilon](t,x)$ of (weak) solutions converge to $[r, \mathbf{V}](t, x_h)$, $\mathbf{V} = [\mathbf{w}, 0]$, where the couple $[r(t,x_h), \mathbf{w}(t,x_h)]$ solves the $2-D$ compressible Navier-Stokes equations on the domain ω :

$$\partial_t r + \mathrm{div}_h(r\mathbf{w}) = 0 \text{ in } (0,T)\times\omega, \tag{17}$$

$$r\partial_t \mathbf{w} + r\mathbf{w}\cdot\nabla_h \mathbf{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h \mathbf{w}) \text{ in } (0,T)\times\omega, \tag{18}$$

$$r(0,x_h) = r_0(x_h),\ \mathbf{w}(0,x_h) = \mathbf{w}_0 := \mathbf{v}_{0,h}(x_h),\ x_h \in \omega, \tag{19}$$

where

$$\mathbb{S}_h(\nabla_h \mathbf{w}) = \mu\Big(\nabla_h \mathbf{w} + (\nabla_h \mathbf{w})^T - \mathrm{div}_h \mathbf{w}\Big) + (\eta + \frac{\mu}{3})\mathrm{div}_h \mathbf{w}\mathbb{I}_h,$$

Our goal is to justify the above (formal) limit in the framework of weak solutions of the *primitive system* (14), (15), (25).
Several geometrical situations :
- Periodic layers.
- Layers over bounded domains ω with no-slip conditions on $\partial\omega$.
- Layers over bounded domains with slip conditions.
- Some particular cases of unbounded layers.

Compressible Navier-Stokes system (14-25) on $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, where $\omega \subset R^2$ is a bounded domain.

$$\mathbf{u}|_{\partial\omega\times(0,\varepsilon)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0. \qquad (20)$$

Rescaling of the equations to a fixed domain.

$$\Omega_\varepsilon \ni (x_h, \varepsilon x_3) \mapsto (x_h, x_3) \in \Omega := \Omega_1, \quad \text{where } x_h = (x_1, x_2), \qquad (21)$$

Here and hereafter, we denote

$$\nabla_\varepsilon = (\nabla_h, \frac{1}{\varepsilon}\partial_{x_3}),\ \nabla_h = (\partial_{x_1}, \partial_{x_2}),$$

$$\mathrm{div}_\varepsilon \mathbf{u} = \mathrm{div}_h \mathbf{v}_h + \frac{1}{\varepsilon}\partial_{x_3} v_3,\ \mathbf{v}_h = (v_1, v_2),\ \mathrm{div}_h \mathbf{v}_h = \partial_{x_1} v_1 + \partial_{x_2} v_2.$$

Denoting the new density and velocity again by ϱ, $\mathbf{u}$, we may rewrite system (14–25) as follows :

$$\partial_t \varrho + \mathrm{div}_\varepsilon(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{22}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_\varepsilon(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_\varepsilon p(\varrho) = \mathrm{div}_\varepsilon \mathbb{S}(\nabla_\varepsilon \mathbf{u}) \quad \text{in } (0,T)\times\Omega, \tag{23}$$

with the boundary conditions

$$\mathbf{u}|_{\partial\omega\times(0,1)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,1\}} = 0. \tag{24}$$

$$\varrho(0,x) = \varrho_{0,\varepsilon}(x) \quad \mathbf{u}(0,x) = \mathbf{u}_{0,\varepsilon}(x), \quad x\in\Omega \tag{25}$$

(where $\varrho_{0,\varepsilon}(x) = \tilde{\varrho}_{0,\varepsilon}(x_h, \varepsilon x_3)$, $\mathbf{u}_{0,\varepsilon}(x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x_h,\varepsilon x_3)$, cf. (**??**)).

Definition

We say that $[\varrho, \mathbf{u}]$ is a finite energy weak solution to the compressible Navier-Stokes (14 - 15) with initial conditions (25) and boundary conditions (24) in the space time cylinder $(0,T)\times\Omega$ if the following holds :

- *the functions $[\varrho, \mathbf{u}]$ belong to the regularity class*

$$\left\{ \begin{array}{c} \varrho \in L^\infty([0,T];L^\gamma(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega,\ \gamma > \frac{3}{2}, \\ \\ \mathbf{u} \in L^2(0,T;W^{1,2}_{\mathbf{n}}(\Omega;R^3)),\ \varrho\mathbf{u}^2 \in L^\infty([0,T];L^1(\Omega)); \end{array} \right\} \tag{26}$$

- *$\varrho \in C_{\rm weak}([0,T];L^\gamma(\Omega))$ and the continuity equation (14) is satisfied in the weak sense,*

$$\int_\Omega \varrho\varphi\ {\rm d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\varphi(0,\cdot)\ {\rm d}x = \int_0^\tau \int_\Omega \varrho\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\ {\rm d}x\ {\rm d}t. \tag{27}$$

for all $\tau \in [0,T]$ and any test function $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$;

Definition

- $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega))$ *and the momentum equation (15) holds in the sense that*

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \,{\rm d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\varphi(0,\cdot)\,{\rm d}x \tag{28}$$

$$= \int_0^T\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_\varepsilon\varphi + p(\varrho_\varepsilon){\rm div}_\varepsilon\varphi \,{\rm d}x\,{\rm d}t$$

$$- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\varphi\,{\rm d}x\,{\rm d}t$$

for all $\tau \in [0,T]$ *and for any* $\varphi \in C_c^\infty([0,T]\times\overline{\Omega}; R^3)$, $\varphi_3|_{\omega\times\{0,1\}} = 0$ *;*

- *the energy inequality*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right](\tau)\,{\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\mathbf{u}\,{\rm d}x\,{\rm d}t \tag{29}$$

$$\le \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon})\right]\,{\rm d}x$$

holds for a.a. $\tau \in (0,T)$

Existence of weak solutions

There exists weak solutions for the above problem with $\mathbb{S}$, p and $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$ verifying assumptions (11), (6), provided the domain ω is Lipschitz. Moreover, any weak solution satisfies below relative entropy inequality (30) with the remainder (31),

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \left(\mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}-\mathbf{U})\Big)\right) : \nabla_\varepsilon(\mathbf{u}-\mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \qquad (30)$$

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) := \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_\varepsilon \mathbf{U}\Big) \cdot (\mathbf{U}-\mathbf{u}) \, \mathrm{d}x \qquad (31)$$

$$+ \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{U}) : \nabla_\varepsilon(\mathbf{U}-\mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \big((r-\varrho)\partial_t H'(r) + \nabla_\varepsilon H'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u})\big) \, \mathrm{d}x - \int_\Omega \mathrm{div}_\varepsilon \mathbf{U} \Big(p(\varrho) - p(r)\Big) \, \mathrm{d}x,$$

where the test functions satisfy

$$r \in C^1([0,T] \times \overline{\Omega}), \ r > 0 \qquad (32)$$

$$\mathbf{U} \in C^1([0,T] \times \overline{\Omega}; R^3), \ \mathbf{U}|_{\partial\omega \times (0,1)} = 0, \ U_3|_{\omega \times \{0,1\}} = 0.$$

Target system and the main result

The expected target system is system (17–19) endowed with the no slip boundary conditions :

$$\mathbf{w}|_{\partial\omega} = 0. \tag{33}$$

Target system Valli, Zajaczkowski

Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$, $\partial\omega \in C^3$ and that

$$r_0 \in W^{2,2}(\omega), \ \inf_{\omega} r_0 > 0, \ \mathbf{w}_0 \in W^{3,2}(\omega; R^2), \tag{34}$$

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big)\Big|_{\partial\omega} = 0. \tag{35}$$

Target system

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_{\omega} r_0 \leq D,$$

then the problem (17–19), (33) admits a unique strong solution (in the sense a.e. in $(0,T) \times \omega$) in the class (36), (37).

$$r \in C([0,T); W^{2,2}(\omega)), \; \mathbf{w} \in C([0,T); W^{2,2}(\omega; R^2)) \cap L^2(0,T; W^{3,2}(\omega; R^2))$$

$$\partial_t r \in C([0,T); W^{1,2}(\omega)), \; \partial_t \mathbf{w} \in L^2(0,T; W^{2,2}(\omega; R^2)). \tag{36}$$

In particular,

$$0 < \underline{r} \equiv \inf_{(t,x_h) \in (0,T) \times \omega} r(t,x_h) \leq \sup_{(t,x_h) \in (0,T) \times \omega} r(t,x_h) \equiv \overline{r}. \tag{37}$$

Main result

Let $\partial\omega \in C^3$, p satisfies hypotheses (6), r_0, $\mathbf{w}_0$ satisfy assumptions (34), (35) and let $T_{\max} > 0$ be the life time of the strong solution to problem (17–19), (33) corresponding to $[r_0, \mathbf{w}_0]$.
Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the $3 - D$ compressible Navier Stokes equations (14-25), (24) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that initial data satisfy

$$\mathcal{E}\left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \Big| r_0, \mathbf{V}_0\right) \to 0, \tag{38}$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.
Then

$$\text{esssup}_{t\in(0,T_{\max})}\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \to 0, \tag{39}$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $(r, \mathbf{w})$ satsifies the $2 - D$ compressible Navier-Stokes system (17–19) with the boundary conditions (33) on the time interval $[0, T_{\max})$.

Corollary

$$\varrho_\varepsilon \to r \text{ strongly in } L^\infty(0,T;L^\gamma(\Omega)),$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{r}\mathbf{V} \text{ strongly in } L^\infty(0,T;L^2(\Omega;R^3)),$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \to r\mathbf{V} \text{ strongly in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)).$$

Main ideas of the proof

-Estimate the left hand side of the relative entropy inequality (30) with test functions $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ from below by

$$\mathcal{E}(\varrho, \mathbf{u} \Big| r, \mathbf{V})(\tau) + c \int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega;R^2)} \mathrm{d}t - c' \int_0^\tau \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \mathrm{d}t \tag{40}$$

-Estimate the right hand side from above by

$$h_\varepsilon(\tau) + \delta \int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega)} \mathrm{d}t + c'(\delta) \int_0^\tau a(t) \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \mathrm{d}t \tag{41}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0, T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0, T).$$

Approximating system of PDEs for weak solutions

Approximating system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0,$$

$$\partial_n \varrho |`_{\partial\Omega} = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \delta \varrho^4) + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}$$

$$= \mu \Delta \mathbf{u} + (\frac{\mu}{3} + \eta) \nabla_x \operatorname{div} \mathbf{u}$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

Weak solutions are obtained letting first $\varepsilon \to 0$ and then $\delta \to 0$. This is not exploitable in the numerics !

Approximation by numerical schemes

Finite element/finite volume set-up Gallouet, Herbin, Eymard, Latché, . . ., since 2009, Karlsen, Karper, since 2010

Convergence proof to the weak equations in the steady Stokes equations (GHEL), weak solutions to the compressible Stokes (KK, GHEL) and Navier-Stokes equations (KK) of several types of numerical schemes, always with limitation $\gamma > 3$. No error estimates are available for the investigated schemes.

Notations

$K \in \mathcal{T}$ - regular partition of Ω into tetrahedrals of size h.

$\sigma = K|L \in \mathcal{E}$ - set of sides of tetrahedrals.

$0 < t_1 < \ldots < t_n < \ldots < T$ - time discretisation of step Δt.

$\varrho(t_n, x) \approx \sum_{K \in \mathcal{T}} \varrho_K^n 1_K(x) \in L_h$ - space of piecewise constants.

$\mathbf{u}(t_n, x) \approx \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}} u_{i,\sigma}^n \phi_\sigma(x) e_i \in W_h = V_h^3$ - the Crouzeix-Raviart finite element space.
Upwind :

$$\varrho_\sigma^{\rm up} = \left\{ \begin{array}{c} \varrho_K \text{ if } \mathbf{u}_\sigma \cdot \mathbf{n}_K > 0 \\ \\ \varrho_L \text{ otherwise} \end{array} \right\}, \quad \text{where } \sigma = K|L.$$

Mean values :

$$\hat{V}_K = \frac{1}{|K|} \int_K V \mathrm{d}x, \quad \hat{V}_\sigma = \frac{1}{|\sigma|} \int_\sigma V \mathrm{d}S$$

Projections

Projections

$$\Pi_h^V : W^{1,p}(\Omega) \to V_h, \quad \Pi_h^V(U) \equiv U_h \equiv \sum_{\sigma \in \mathcal{E}} \hat{U}_\sigma \phi_\sigma$$

$$\Pi_h^L : L^p(\Omega) \to L_h, \quad \Pi_h^L(r) \equiv r_h \equiv \sum_{K \in \mathcal{T}} \hat{r}_K 1_K$$

Estimates involving projections

Let $s = 1, 2$, $1 \leq p \leq \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall r \in W^{1,p}(K), \quad \|r_h - r\|_{L^p(K)} \leq ch\|\nabla_x r\|_{L^p(K)},$$

$$\forall U \in W^{s,p}(K), \quad \|U_h - U\|_{L^p(K)} \leq ch^s\|\nabla_x^s U\|_{L^p(K)}$$

$$\forall U \in W^{s,p}(K), \quad \|\nabla_x U_h - \nabla_x U\|_{L^p(K)} \leq ch^{s-1}\|\nabla_x^s U\|_{L^p(K)}.$$

Some auxiliary estimates

Poincaré type inequalities

Let $1 \le p \le \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_{h,K}\|_{L^p(K)} \le ch\|\nabla_x U_h\|_{L^p(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}$$

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_\sigma\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}.$$

Sobolev type inequalities

Let $2 \le p \le 6$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x V\|_{L^2(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_\sigma\|_{L^p(K)} \le h^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x U\|_{L^p(K)}$$

Numerical scheme

$$|K|\frac{\varrho_K^n-\varrho_K^{n-1}}{\Delta t}+\sum_{\sigma\in\mathcal{E}_K}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\mathbf{u}_\sigma\cdot\mathbf{n}_K=0,$$

$$\sum_{K\in\mathcal{T}}|K|\frac{\varrho_K^n\hat{u}_{i,K}^n-\varrho_K^{n-1}\hat{u}_{i,K}^{n-1}}{\Delta t}\hat{\phi}_{\sigma',K}+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{u}_{i,K,\sigma}^{n,\mathrm{up}}\mathbf{u}_\sigma\cdot\mathbf{n}_K\hat{\phi}_{\sigma',K}\mathrm{d}S$$

$$-\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}|\sigma|p(\varrho_K^n)n_{i,K}\delta_{\sigma,\sigma'}$$

$$+\mu\sum_{K\in\mathcal{T}}\int_K\nabla_x\mathbf{u}^n:(\nabla_x\phi_{\sigma'}\otimes\mathbf{e}_i)\mathrm{d}x+(\frac{\mu}{3}+\eta)\sum_{K\in\mathcal{T}}\int_K\mathrm{div}\mathbf{u}^n:\partial_i\phi_{\sigma'}\mathrm{d}x=0,$$

$$\sigma'\in\mathcal{E},\quad i=1,2,3$$

$$\mathbf{u}_\sigma=0\quad\text{if } \sigma\in\mathcal{E}_{\mathrm{ext}}.$$

This a non linear algebraic equation. Existence of a solution

$$(\varrho_K>0,u_{i,\sigma})_{K\in\mathcal{T},\sigma\in\mathcal{E},i=1,2,3}$$

follows from the topological degree fixed point theory.

Energy inequality - continuous case

"Continuous" energy inequality

$$\int_\Omega \frac{1}{2}\varrho \mathbf{u}^2 \mathrm{d}x\Big|_0^\tau + \int_0^\tau \int_\Omega \mu |\nabla_x \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t \le 0$$

Energy inequality - discrete case

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n|^2 - \varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1}|^2\Big) + \sum_K \frac{|K|}{\Delta t}\Big(H(\varrho_K^n) - H(\varrho_K^{n-1})\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\varrho_K^{n-1}\frac{|\hat{\mathbf{u}}_K^n - \hat{\mathbf{u}}_K^{n-1}|^2}{2} + \sum_K \frac{|K|}{\Delta t}H''(\varrho_K^{n-1,n})\frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} \frac{1}{4}|\sigma|\varrho_\sigma^{n,\mathrm{up}}(\hat{\mathbf{u}}_K^n - \hat{\mathbf{u}}_L^n)^2\ |\mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K}|$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} \frac{1}{4}|\sigma|H''(\varrho_{KL}^n)\Big(\varrho_K^n - \varrho_L^n\Big)^2\ |\mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K}|$$

$$+\sum_K \Big(\mu\int_K |\nabla_x \mathbf{u}|^2 \mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K |\mathrm{div}\mathbf{u}|^2\mathrm{d}x\Big) \leq 0.$$

Discrete relative energy

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n - \hat{\mathbf{U}}_{h,K}^n|^2 - \varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1} - \hat{\mathbf{U}}_{h,K}^{n-1}|^2\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\Big(E(\varrho_K^n|\hat{r}_K^n) - E(\varrho_K^{n-1}|\hat{r}_K^{n-1})\Big)$$

$$+\sum_K \Big(\mu\int_K |\nabla_x(\mathbf{u}^n - \mathbf{U}_h^n)|^2 \mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K |\mathrm{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 \mathrm{d}x\Big)$$

$$\preceq \sum_K \Big(\mu\int_K \nabla_x\mathbf{U}_h : \nabla_x(\mathbf{U}_h - \mathbf{u})\mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K \mathrm{div}\mathbf{U}_h \mathrm{div}(\mathbf{U}_h - \mathbf{u})\mathrm{d}x\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\Big(\varrho_K^{n-1}(\hat{\mathbf{U}}_{h,K}^{n-1} - \hat{\mathbf{u}}_{h,K}^{n-1})\cdot(\hat{\mathbf{U}}_{h,K}^n - \hat{\mathbf{U}}_{h,K}^{n-1}) + (\hat{r}_K^n - \varrho_K)(H'(\hat{r}_K^n) - H'(\hat{r}_K^{n-1})\Big)$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}}\Big)\cdot\hat{\mathbf{U}}_{h,K}(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

$$-\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| p(\varrho_K)(\hat{\mathbf{U}}_{h,\sigma}^n\cdot\mathbf{n}_{\sigma,K}) - \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}} H'(\hat{r}_K^n)(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

Formulation of the error estimates

Discrete relative energy functional

$$\mathcal{E}\left(\varrho^n, \mathbf{u}^n \middle| r, U\right) = \sum_{K\in\mathcal{T}} \frac{1}{2}\varrho_K^n(\hat{\mathbf{U}}_{h,K}^n - \hat{\mathbf{u}}_K^n)^2 + E(\varrho_K^n|\hat{r}_K^n)$$

Example of an error estimate

Let e.g. $p(\varrho) = \varrho^\gamma$ and $\gamma > 3/2$. Let (r, U) be a strong solution of the compreessible Navier-Stokes equations emanating from the finite energy intial data (r_0, U_0). Let $(\varrho, \mathbf{u})$ be the KK approximation of the compressible Navier system emanating from the initial data $(\varrho_0|\mathbf{u}_0)$. Then there exists $c > 0$ independent of h, Δt, ϱ, $\mathbf{u}$, such that

$$\mathcal{E}\left(\varrho^n, \mathbf{u}^n \middle| r, U\right) \leq c\Big(\mathcal{E}\left(\varrho_0, \mathbf{u}_0 \middle| r_0, \mathrm{U}_0\right) + h^\alpha + \Delta t\Big),$$

where

$$\alpha = \left\{ \begin{array}{c} \frac{2\gamma-3}{2} \text{ if } 3/2 < \gamma < 2 \\ \frac{1}{2} \text{ if } \gamma \geq 2 \end{array} \right\}$$

Sketch of the proof : Treatment of the Red term

$$\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}}\right) \cdot \hat{\mathbf{U}}_{h,K} (\mathbf{u}^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}}\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)(\mathbf{u}^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,K}^{n} - \hat{\mathbf{u}}_K^{n}\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)(\mathbf{u}^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_K^{n} \left(\hat{\mathbf{U}}_{h,K}^{n} - \hat{\mathbf{u}}_K^{n}\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)(\mathbf{u}^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_K^{n} \left(\hat{\mathbf{U}}_{h,K}^{n} - \hat{\mathbf{u}}_K^{n}\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)\left(\mathbf{u}_\sigma^n - \mathbf{U}_\sigma^n\right) \cdot \mathbf{n}_{\sigma,K}$$

$$+ \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_K^{n} \left(\hat{\mathbf{U}}_{h,K}^{n} - \hat{\mathbf{u}}_K^{n}\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)\mathbf{U}_\sigma \cdot \mathbf{n}_{\sigma,K}$$

$$\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_K^n \left(\hat{\mathbf{U}}_{h,K}^n - \hat{\mathbf{u}}_K^n\right) \cdot (\hat{\mathbf{U}}_{h,K} - \mathbf{U}_\sigma)\mathbf{U}_\sigma \cdot \mathbf{n}_{\sigma,K}$$

$$\sum_K \int_K r\partial_t \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}_h) + \sum_K \int_K r\mathbf{U} \cdot \nabla\mathbf{U} \cdot (\mathbf{u} - \mathbf{U}_h) + \ldots = \ldots$$

$$\sum_K \int_K r\mathbf{U} \cdot \nabla\mathbf{U} \cdot (\mathbf{u} - \mathbf{U}_h) \approx \sum_K \int_K \hat{r}_K \hat{\mathbf{U}}_{h,K} \cdot \nabla\mathbf{U} \cdot (\hat{\mathbf{u}}_K - \hat{\mathbf{U}}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \hat{r}_K \hat{\mathbf{U}}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U} - \hat{\mathbf{U}}_{h,K}) \cdot (\hat{\mathbf{u}}_K - \hat{\mathbf{U}}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \hat{r}_K \hat{\mathbf{U}}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U}_\sigma - \hat{\mathbf{U}}_{h,K}) \cdot (\hat{\mathbf{u}}_K - \hat{\mathbf{U}}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \hat{r}_K (\mathbf{U}_\sigma - \hat{\mathbf{U}}_{h,K}) \cdot (\hat{\mathbf{u}}_K - \hat{\mathbf{U}}_{h,K}) \hat{\mathbf{U}}_\sigma \cdot \mathbf{n}_{\sigma,K}$$