# Existence of Weak Solutions for Model of Chemically Reacting Mixture

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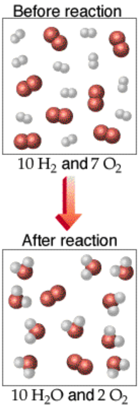
# Example: the $H_2$-$O_2$ system




The initiation reactions are

$$H_2 + M \rightarrow H + H + M \quad \text{(very high temperatures)} \tag{H.1}$$

$$H_2 + O_2 \rightarrow HO_2 + H \quad \text{(other temperatures).} \tag{H.2}$$

Chain-reaction steps involving O, H, and OH radicals are

$$H + O_2 \rightarrow O + OH, \tag{H.3}$$

$$O + H_2 \rightarrow H + OH, \tag{H.4}$$

$$H_2 + OH \rightarrow H_2O + H, \tag{H.5}$$

$$O + H_2O \rightarrow OH + OH. \tag{H.6}$$

Chain-terminating steps involving O, H, and OH radicals are the three-body recombination reactions:

$$H + H + M \rightarrow H_2 + M, \tag{H.7}$$

$$O + O + M \rightarrow O_2 + M, \tag{H.8}$$

$$H + O + M \rightarrow OH + M, \tag{H.9}$$

$$H + OH + M \rightarrow H_2O + M. \tag{H.10}$$

To complete the mechanism, we need to include reactions involving $HO_2$, the hydroperoxy radical, and $H_2O_2$, hydrogen peroxide. When

$$H + O_2 + M \rightarrow HO_2 + M \tag{H.11}$$

becomes active, then the following reactions, and the reverse of H.2, come into play:

$$HO_2 + H \rightarrow OH + OH, \tag{H.12}$$

$$HO_2 + H \rightarrow H_2O + O, \tag{H.13}$$

$$HO_2 + O \rightarrow O_2 + OH, \tag{H.14}$$

and

$$HO_2 + HO_2 \rightarrow H_2O_2 + O_2, \tag{H.15}$$

$$HO_2 + H_2 \rightarrow H_2O_2 + H, \tag{H.16}$$

with

$$H_2O_2 + OH \rightarrow H_2O + HO_2, \tag{H.17}$$

$$H_2O_2 + H \rightarrow H_2O + OH, \tag{H.18}$$

$$H_2O_2 + H \rightarrow HO_2 + H_2, \tag{H.19}$$

$$H_2O_2 + M \rightarrow OH + OH + M. \tag{H.20}$$

S.R. Turns "*An Introduction to combustion*"

# The system

One of possible formulations

$$
\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,\\
&\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \operatorname{div} \mathbf{S} + \nabla p = \mathbf{0},\\
&\partial_t(\varrho E) + \operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div} \mathbf{Q} + \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbf{S}\mathbf{u}) = 0,\\
&\partial_t(\varrho Y_k) + \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k = \omega_k, \quad k = 1, \dots n,
\end{aligned}
\tag{1}
$$

where

$$\varrho E = \frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e$$

and $Y_k \geq 0$ are defined by $Y_k = \frac{\varrho_k}{\varrho}$ and they satisfy $\sum_{k=1}^n Y_k = 1$.

V. Giovangigli: Multicomponent Flow Modelling 1999.

# The entropy – compatibility with the II Law of Thermodynamics

The total derivatives of $Y_k$, $\varrho^{-1}$ and $e$ are related by the Gibbs formula

$$\vartheta \mathbf{D}s = \mathbf{D}e + p\mathbf{D}\varrho^{-1} - \sum_{k=1}^{n} g_k \mathbf{D} Y_k,$$

where s denotes the entropy and is satisfies the equation

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta} \mathbf{F}_k\right) = \sigma,$$

and the entropy production rate $\sigma$ equals

$$\sigma = \underbrace{\frac{(2\mu \mathbf{D}(\mathbf{u}) + \nu \operatorname{div} \mathbf{u} \mathbf{I}) : \nabla \mathbf{u}}{\vartheta}}_{\geq 0} + \underbrace{\frac{\kappa |\nabla \vartheta|^2}{\vartheta^2}}_{\geq 0} \underbrace{- \sum_{k=1}^{n} \frac{g_k \omega_k}{\vartheta}}_{\substack{\textit{admissibility} \\ \textit{condition} \\ \geq 0}} \underbrace{- \sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla (\log p_k)}_{\substack{\textit{what if} \\ F_k \approx -\nabla Y_k \ ?}}.$$

# General diffusion

$$\mathbf{F}_k = -\sum_{l=1}^{n} C_{kl}\mathbf{d}_l, \qquad \mathbf{d}_k = \nabla\left(\frac{p_k}{p}\right) + \left(\frac{p_k}{p} - \frac{\varrho_k}{\varrho}\right)\nabla \log p,$$

$$p = \sum_{k=1}^{n} p_k = R\varrho\vartheta \sum_{k=1}^{n} \frac{Y_k}{m_k}, \qquad \sum_{k=1}^{n} \mathbf{F}_k = 0, \quad \sum_{k=1}^{n} \mathbf{d}_k = 0.$$

The main properties of the flux diffusion matrix $C$ are:

$$C\mathcal{Y} = \mathcal{Y}C^{\boldsymbol{T}}, \quad N(C) = \text{lin}\{\vec{Y}\}, \quad R(C) = U^{\perp},$$

where

$\mathcal{Y} = \text{diag}(Y_{\mathbf{1}}, \ldots, Y_{\boldsymbol{n}})$, $\vec{Y} = (Y_{\mathbf{1}}, \ldots, Y_{\boldsymbol{n}})^{\boldsymbol{T}}$

$N(C)$ - the nullspace of $C$, $R(C)$ - the range of $C$,

$\vec{U} = (1, \ldots, 1)^{\boldsymbol{T}}$, $U^{\perp}$ - the orthogonal complement of $\text{lin}\{\vec{U}\}$.

In general $C$ is not symmetric and not positive-definite, but it guarantees that

$$-\sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla\left(\log p_k\right) = \frac{p}{\vartheta} \sum_{k,l=1}^{n} D_{kl}\mathbf{d}_k\mathbf{d}_l \geq 0.$$

# An example

→ If $D$ is given by

$$\varrho D = \begin{pmatrix} \frac{Z_1}{Y_1} & -1 & \dots & -1 \\ -1 & \frac{Z_2}{Y_2} & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & \frac{Z_n}{Y_n} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & 0 & 1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{on } U^{\perp},$$

where $Z_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} Y_i$

→ then $C_{kl} = \varrho Y_k D_{kl}$ has the following form

$$C = \begin{pmatrix} Z_1 & -Y_1 & \dots & -Y_1 \\ -Y_2 & Z_2 & \dots & -Y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -Y_n & -Y_n & \dots & Z_n \end{pmatrix}.$$

# The Maxwell-Stefan equations

$$\partial_t(\varrho Y_k) + \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div}\mathbf{F}_k = \omega_k(\varrho, \vartheta, Y_1, \ldots, Y_n),$$

$$\mathbf{F}_i = \varrho Y_i \mathbf{V}_i, \qquad \sum_{k=1}^{n} Y_i \mathbf{V}_i = 0, \quad \mathbf{V}_i = -\sum_{k=1}^{n} D_{ik}\mathbf{d}_k$$

The diffusion velocities are given (implicitly) in terms of the gradients of the mole fractions $X_i = \frac{p_i}{p}$ by the Stefan-Maxwell equations

$$\overbrace{\nabla X_i - \underbrace{(Y_i - X_i)\left(\frac{\nabla p}{p}\right)}_{small}}^{\mathbf{d}_i} = \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\frac{X_i X_j}{\mathcal{D}_{ij}}\right)(\mathbf{V}_j - \mathbf{V}_i),$$

where $\mathcal{D}_{ij} > 0$ denotes the binary diffusion coefficient, $\mathcal{D}_{ij} = \mathcal{D}_{ji}$.

Local in time well posedness Bothe '10. Global existence of solutions and exponential decay to the homogeneous steady state Jüngel, Stelzer '12.

Remark: Our example corresponds to the case $\mathcal{D}_{ij} m_i m_j = const$.

# How to meet Navier-Stokes?

- Data sufficiently close to an equilibrium state
  - Global existence in time and asymptotic stability of constant stationary states, Giovangigli '99.
- Simplified mixing aspect
  - Weak solutions for one fluid: Lions '98, Feireisl, Novotný, Petzeltová '01, Veigant, Kazhikhov '95 (2D case, $\nu(\varrho) = c\varrho^{\alpha}$, $c > 0$, $\alpha \geq 3$).
  - Weak variational solutions for mixtures with

$$p = \varrho^{\gamma} + \varrho\vartheta, \quad \mathbf{F}_{k} = -D\nabla Y_{k}, \quad k = 1, \ldots, n,$$

    Feireisl, Petzeltová, Trivisa '08.
  - Large data existence for multifluids models: Frehse, Goj, Málek '05, Mamontov, Prokudin '12.
- "Incompressiblity condition" $\operatorname{div} \mathbf{u} = 0$
  - Maxwell-Stefan coupled to incompressible Navier-Stokes: Chen, Jüngel '13, including heat conduction Marion, Temam '13, including thermal diffusion cross effects Bulíček, Havrda '13.

# Reaction-diffusion equations ($\vartheta = const$)

## Theorem (Mucha, Pokorný, Zatorska '13)

*Let $\varrho$ be a sufficiently smooth solution to*

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad \varrho^0 = \sum_{k=1}^{n} \varrho_k^0,$$

*such that* $\sqrt{\varrho} \in L^2(0, T; W^{1,2}(\Omega))$; $0 < \inf_\Omega \varrho^0 \le \sup_\Omega \varrho^0 < \infty$; $\mathbf{u} \in L^\infty((0, T) \times \Omega)$; $\operatorname{div} \mathbf{u} \in L^2((0, T); L^\infty(\Omega))$; $\int_\Omega \sum_{k=1}^n \frac{\log p_k \omega_k \varrho}{m_k} \, \mathrm{d}x \le c$.

*Then, the system*

$$\partial_t \varrho_k + \operatorname{div}(\varrho_k \mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho \omega_k, \quad k = 1, \ldots n$$

*admits a global in time weak solution, such that*

$$\varrho_k \ge 0 \quad \textit{a.e. in } (0, T) \times \Omega, \quad \sum_{k=1}^{n} \int_\Omega \varrho_k(t) \, \mathrm{d}x = M_0.$$

*Furthermore, the following regularity properties hold:*

$$\varrho_k \in C([0, T]; L \log L(\Omega)) \quad \textit{and} \quad \nabla \sqrt{\varrho_k} \in L^2((0, T) \times \Omega).$$

# How to start the proof?

We introduce the entropy variables

$$r_k = \log p_k$$

and consider the following approximate system:

$$(\delta + e^{r_k})\partial_t r_k + \mathrm{div}(e^{r_k}\mathbf{u}) - \mathrm{div}\left((\delta + \varepsilon e^{r_k})\nabla r_k\right) + \frac{\mathrm{div}\,\mathbf{F}_k}{m_k} = \frac{\varrho\omega_k}{m_k},$$

$$r_k(0,x) = r_k^0.$$

The basic a priori estimate

$$\sup_{t\in(0,T)} \sum_{k=1}^{n} \| \left(\delta r_k^2 + e^{r_k} r_k\right)(t)\|_{L^1(\Omega)}$$

$$+\sum_{k=1}^{n}\left\{ \delta \int_0^T \|\nabla r_k\|_{L^2(\Omega)}\,\mathrm{d}t + \varepsilon \int_0^T \|\nabla\sqrt{e^{r_k}}\|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T \left\|\frac{\mathbf{F}_k}{\sqrt{m_k e^{r_k}}}\right\|^2_{L^2(\Omega)}\,\mathrm{d}t \right\} \le c.$$

# Pressure decomposition

Denoting $C\nabla_{x_i} p = (\nabla_{x_i} p)^I$, where

$$p = (p_1, \dots, p_n)^T \quad \text{and} \quad \nabla p = (\nabla p_1, \dots, \nabla p_n)^T$$

we may decompose each $(k,i)$-th coordinate $k \in \{1, \dots, n\}, i \in \{1,2,3\}$

$$(\nabla_{x_i} p)_k = (\nabla_{x_i} p)^I_k + \alpha_i Y_k \quad \Big| \cdot m_k \text{ and } \sum_{k=1}^n$$

$$\alpha_i = \frac{\nabla_{x_i} \varrho}{\sum_{k=1}^n m_k Y_k} - \frac{\sum_{k=1}^n m_k (\nabla_{x_i} p)^I_k}{\sum_{k=1}^n m_k Y_k}.$$

So, the full gradients of partial pressures can be expressed in terms of gradients of density and the gradient of "known" part of the pressure

$$\nabla p = (\nabla p)^I + \left( \frac{\nabla \varrho}{\sum_{k=1}^n m_k Y_k} - \frac{\sum_{k=1}^n m_k (\nabla p)^I_k}{\sum_{k=1}^n m_k Y_k} \right) Y.$$

# Isothermal reversible reaction $A \rightleftharpoons B$

$$\left.\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu \mathbf{D}(\mathbf{u})) - \nabla(\nu \operatorname{div} \mathbf{u}) + \nabla \pi = \mathbf{0} \\ \partial_t(\varrho Y_A) + \operatorname{div}(\varrho Y_A \mathbf{u}) + \operatorname{div}(\mathbf{F}_A) = \varrho \omega \end{aligned}\right\} \quad \text{in } (0,T) \times \mathbb{T}^3$$

with variable pressure

$$\pi = \underbrace{\varrho^\gamma}_{\pi_c} + \underbrace{\frac{\varrho Y_A}{m_A} + \frac{\varrho Y_B}{m_B}}_{\pi_m}, \qquad \gamma > 1,$$

general diffusion flux and viscosity vanishing on vacuum

$$\mathbf{F}_A = -\left(\nabla p_A - Y_A \nabla \pi_m\right), \qquad \mathbf{F}_B = -\mathbf{F}_A,$$
$$\nu(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho), \quad (\text{e.g. } \mu(\varrho) = \varrho,\ \nu(\varrho) = 0)$$

Z.: *On the flow of chemically reacting gaseous mixture* '12.

Remark: The unknown functions are $(\varrho, \sqrt{\varrho}\mathbf{u}, \sqrt{\varrho} Y_A)$ due to possible vacuum!

# A-priori estimates (1/3)

1. Comparison principle:

$$0 \leq Y_A \leq 1 \quad \text{on } \Omega \times (0, T).$$

2. Energy estimate:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\gamma - 1} \varrho^\gamma - \frac{1}{m_B} \varrho \log \varrho \right) \, \mathrm{d}x + \int_\Omega 2\mu(\varrho) |\mathbf{D}(\mathbf{u})|^2 \, \mathrm{d}x + \int_\Omega \nu(\varrho) (\operatorname{div} \mathbf{u})^2 \, \mathrm{d}x \leq c.$$

3. Comparison principle for the gradient of concentration of species $A$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \frac{1}{2} \varrho Y_A^2 \, \mathrm{d}x + \frac{1}{\max\{m_A, m_B\}} \int_\Omega \varrho |\nabla Y_A|^2 \, \mathrm{d}x \leq c(m_A, m_B, \|Y_A\|_\infty, \|\omega\|_\infty) \left( 1 + \int_\Omega |\nabla \varrho \cdot \nabla Y_A| \, \mathrm{d}x \right).$$

# A-priori estimates (2/3)

4. B-D "entropy" estimate

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u} + \nabla \phi(\varrho)|^2 + (\ldots) \right) \mathrm{d}x + \frac{1}{2} \int_\Omega \mu(\varrho) |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 \, \mathrm{d}x + \int_\Omega \nabla \phi(\varrho) \cdot \nabla \pi_c(\varrho) \, \mathrm{d}x$$
$$\leq c(m_A, m_B, \|Y_A\|_\infty, \|\sqrt{\varrho}\, \mathrm{div}\, \mathbf{u}\|_2)$$

for $\phi$ such that $\nabla \phi(\varrho) = 2 \frac{\mu'(\varrho) \nabla \varrho}{\varrho}$.

D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, andshallow water systems (2003)

# A-priori estimates (3/3)

5. Additional estimate for the velocity field

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \frac{1}{2}\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\,\mathrm{d}x + \frac{c}{2}\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x$$
$$\leq c\left(\int_\Omega\left(\frac{\pi(\varrho,Y)^2\varrho^{-\frac{\delta}{2}}}{\mu(\varrho)}\right)^{\frac{2}{2-\delta}}\mathrm{d}x\right)^{\frac{2-\delta}{2}}\left(\int_\Omega \varrho\left(2+\ln(1+|\mathbf{u}|^2)\right)^{\frac{2}{\delta}}\mathrm{d}x\right)^{\frac{\delta}{2}}$$
$$+\,c\int_\Omega \mu(\varrho)|\nabla\mathbf{u}|^2\,\mathrm{d}x$$

we multiply the momentum equation by $(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}$.

A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations (2007).

Assumptions:

- The diffusion fluxes

$$\mathbf{F}_A = -\frac{C_0}{\pi_m}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_A - \frac{\varrho_A}{\varrho m_B}\nabla\varrho\right),$$
$$\mathbf{F}_B = -\frac{C_0}{\pi_m}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_B - \frac{\varrho_B}{\varrho m_A}\nabla\varrho\right).$$

- The viscosity coefficients

$$\mu(\varrho) = \varrho, \quad \nu(\varrho) = 0.$$

- The " cold " component of the internal pressure

$$\pi_c'(\varrho) = \begin{cases} \varrho^{-4k-1} & \text{for } \varrho \le \varrho^*,\ k > 1, \\ \varrho^{\gamma-1} & \text{for } \varrho > \varrho^*,\ \gamma > 1. \end{cases}$$

- The molar production rate

$$-\underline{\omega} \le \omega(Y_A, Y_B) \le \overline{\omega}, \quad \text{and} \quad \omega(Y_A, Y_B) \ge 0 \quad \text{for} \quad Y_A = 0.$$

## Existence of solutions

**Theorem (Mucha, Pokorný, Z. '13)**

*Let $\Omega = \mathbb{T}^3$ and the structural properties be satisfied; the initial data $\varrho^0, \mathbf{u}^0, \varrho_A^0$ satisfy all the natural bounds and*

$$\int_\Omega \left( \frac{1}{2} \frac{|(\varrho \mathbf{u})^0|^2}{\varrho^0} + \pi_c\left(\varrho^0\right) \right) \mathrm{d}x < \infty, \quad \int_\Omega \frac{|\nabla \varrho^0|^2}{\varrho^0} \mathrm{d}x < \infty.$$

*Then there exists a global in time weak solution, s.t.:*

$$\varrho \in L^\infty(0,T; L^1 \cap L^\gamma(\Omega)), \quad \underline{\sqrt{\varrho} \in L^\infty(0,T; H^1(\Omega))}, \quad \varrho > 0 \text{ a.e. on } (0,T) \times \Omega,$$

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T; L^2(\Omega)), \quad \sqrt{\varrho}\nabla\mathbf{u} \in L^2(0,T; L^2(\Omega)), \quad \mathbf{u} \in L^p(0,T; L^{q^*}(\Omega)),$$

$$\underline{\sqrt{\varrho}\nabla Y_A \in L^2(0,T; L^2(\Omega))}, \quad 0 \le Y_A \le 1 \text{ a.e. on } (0,T) \times \Omega,$$

*where* $p = \frac{8k}{4k+1}, \ q^* = \frac{24k}{4k+1}, \ k > 1.$

# Strategy of the proof (1/3)

1. We introduce the $n$-dimensional Faedo-Galerkin approximation for the momentum equation, truncations of coefficients in the equations of species and additional five parameters $\kappa_1, \kappa_2, \varepsilon, \eta$ and $\delta$

$$(\star)\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\varrho \mathbf{D}(\mathbf{u})) + \nabla \pi_{\kappa_2}(\varrho, \varrho_A, \varrho_B) \\ \qquad -\delta \varrho \nabla \left(\Delta^{2s+1} \varrho\right) + \eta \Delta^2 \mathbf{u} + \varepsilon(\nabla \varrho \cdot \nabla)\mathbf{u} = \mathbf{0} \\ \partial_t \varrho_A - \varepsilon \Delta \varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) - \operatorname{div} \mathbf{F}^+_{A,\kappa_1} = \varrho \left(\omega\left(\frac{\varrho_A}{\varrho}\right)\right)_{\kappa_1} \\ \partial_t \varrho_B - \varepsilon \Delta \varrho_B + \operatorname{div}(\varrho_B \mathbf{u}) - \operatorname{div} \mathbf{F}^+_{B,\kappa_1} = -\varrho \left(\omega\left(\frac{\varrho_A}{\varrho}\right)\right)_{\kappa_1}. \end{cases}$$

The goal is to let $n \to \infty$, $\kappa_1, \kappa_2, \varepsilon, \eta, \delta \to 0$ and to prove $\varrho_A + \varrho_B = \varrho$.

# Strategy of the proof (2/3)

2. We linearize system $(\star)$.

3. We set $\mathbf{u} \in C([0,T];X_n)$ for which we find the mappings

$$\mathbf{u} \mapsto \varrho(\mathbf{u}) \quad \text{and} \quad \mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$$

   determining the unique solution to the continuity equation and the species mass balance equations.

4. For sufficiently small time interval $[0, \tau^0]$ we find the unique solution to the momentum equation applying the Banach fixed point theorem. Then we extend the existence result for the maximal time interval.

5. We recover the semi-linear system $(\star)$ using a version of the Leray-Schauder fixed point theorem.

# Strategy of the proof (3/3)

6. Derivation of the energy estimate; passage to the limit $\kappa_1 \to 0$.

7. We test the species equations by $(\varrho_A^- + l)^{q-1}$, $l > 0$, $q \in (1, 2]$

$$\varrho_A^- = \begin{cases} -\varrho_A & \text{if} \quad \varrho_A < 0, \\ 0 & \text{if} \quad 0 \leq \varrho_A, \end{cases}$$

   and let $l \to 0^+$, $q \to 1^+$ in order to get rid of truncations.

8. Passage to the limit with $\kappa_2 \to 0$, $n \to \infty$ (we lose possibility of testing the momentum equation by the solution).

9. Derivation of the B-D entopy, and passage to the limit with $\eta, \varepsilon \to 0$ and $\delta \to 0$, subsequently.

# Perspectives

- Removing the singular pressure, generalization of the diffusion matrix.
- Complete scheme of approximation of weak solutions:
  $\rightarrow$ Mucha, Pokorný, Z. '14 using the weak (energy) formulation.
- What is an incompressible mixture?
- Existence of strong solutions (for short time) and weak-strong uniqueness.