On the Kelvin-Helmholtz Instability with a free surface

D. BRESCH

Laboratoire de Mathématiques, UMR5127 CNRS
Université de Savoie
73376 Le Bourget du lac
France
Email: didier.bresch@univ-savoie.fr

Mathématiques Fluides
Thanks for the invitation to C. Galusinski, F. Golay, A. Novotny

Main Goal

Simple mathematical results linked to Kelvin-Helmoltz instabilities with free surface

Joint work 2011 (published ZAMP 2013) with M. RENARDY (Virginia Tech, USA)

Remark: Be careful with the amazing notations which will be used......
Change g for physical purposes by $1/Fr^2$.
Denote L the ratio between the bottom layer and the upper layer depths.

Many studies concern parallel shear flows between parallel plates.

Here : Shear flow with a free surface at the top and a wall on the bottom.

While the wall bounded shear flow is unstable to all wave lengths, free surface flows without gravity have no such instabilities (see papers by M. Renardy).

The situation of free surface flows with gravity interpolates between the two case : If Froude number tends to 0, the upper surface is effectively rigid.

Starting point :
In a mathematical paper (*Comm. Math. Phys.*, 2008) authors claimed to prove: All monotone profiles with inflection points have long wave instabilities.

Numerical evidence, this is not true : M. and Y. Renardy (*J. Math Fluid Mech* 2011).

Mathematical proof in the long wave regime (the goal of this talk):
– Monotone profiles always: Long wave stability for large enough Froude number.
– An example which admits long wave instabilities for all finite Froude number.

$\Longrightarrow$ Errata by the *Comm. Math. Phys.*'s authors.

Numerical calculations by M. and Y. Renardy (*J. Math. Fluid Mech*):

Examples:
1– Poiseuille flow $U(y) = 1 - y^2$,
2– Hyperbolic tangent shear layer $U(y) = \tanh(ky)$ with $k = 1/2, k = 2$,
3– Cubic profile $U(y) = y + y^3/2$.

1– No inflection point, a velocity extremum.
2– Inflection point at 0 with U and U'' opposite signs.
3– Inflection point at 0 with U and U'' same signs.

Neutral limiting modes (which values for c in the limit $Im(c) \to 0$):
– In wall bounded case: wave speed equal to inflexion value of the base profile.
– In the free surface case: wave speed may also equal to
either the velocity at the bottom or an extremum value of the velocity.

Physical papers:

– J.C. Burns (*Proc. Camb. Phil. Soc.*, 1953),

– A.G. Voronovich, E.D. Lobanov, S.A. Rybak (*Izv. Atmos. Ocean Phys.*, 1980),

– M.S. Longuet-Higgins (*J. Fluid Mech*, 1998),

– M.A. Bakas, P.J. Ioannou (*Phys. of fluids*, 2009)

with a piecewice linear profile.

Let us consider the following system:

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= 0, \\ v_t + uv_x + vv_y + p_y + g &= 0, \\ u_x + v_y &= 0, \end{aligned} \quad (1)$$

in the domain $-1 < y < 1 + h(x,t)$. The lower boundary is a wall,

$$v(x,-1,t) = 0, \quad (2)$$

and the upper boundary is a free surface,

$$p(x, 1 + h(x,t), t) = 0. \quad (3)$$

We also have the kinematic condition

$$h_t + u(x, 1 + h(x,t), t)h_x = v(x, 1 + h(x,t), t). \quad (4)$$

Standard linearization around

$$h = 0, \quad v = 0, \quad p = g(1-y), \quad u = U$$

and perturbation under the form $\exp(i\alpha(x-ct))$.

$$\begin{aligned} i\alpha(U-c)u + U'v + i\alpha p &= 0, \\ i\alpha(U-c)v + p' &= 0, \\ i\alpha u + v' &= 0, \end{aligned} \tag{5}$$

subject to the boundary conditions

$$v(-1) = 0, \qquad p(1) = gh, \qquad i\alpha(U-c)h = v(1). \tag{6}$$

By eliminating variables between the equations, we obtain a single equation for p:

$$\left(-\frac{p'}{(U-c)^2}\right)' + \alpha^2 \frac{p}{(U-c)^2} = 0. \tag{7}$$

The boundary conditions reduce to

$$p'(-1) = 0, \qquad p(1) = \frac{g p'(1)}{\alpha^2 (U-c)^2}. \tag{8}$$

Numerical results by M. and Y. RENARDY

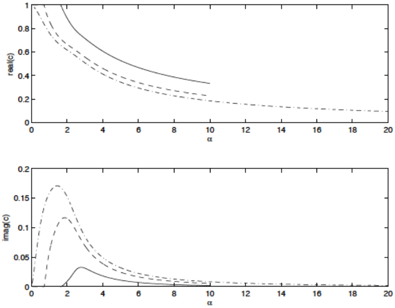

FIGURE 1. Unstable eigenvalues for Poiseuille flow. $g = 0$: -.-, $g = 0.1$: − − −, $g = 0.5$: solid line.

With increasing g, we have an increase of $\alpha_{\min}$ and diminishing growth rate. The wave speed, i.e. the real part of c, approaches the velocity maximum 1 at $\alpha_{\min}$ and the surface velocity 0 at infinity.

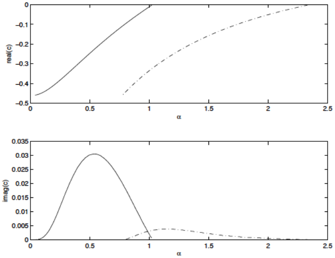

FIGURE 2. Unstable eigenvalues for hyperbolic tangent profile $U = \tanh(y/2)$. $g = 0$: solid line, $g = 0.3$: -.-.

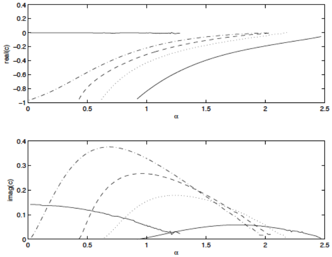

FIGURE 3. Unstable eigenvalues for hyperbolic tangent profile $U = \tanh(2y)$. $g = 0$: -.-, $g = 0.5$: – – –, $g = 1$: ..., $g = 2$: solid line.

In the wall bounded case, this flow is known to be stable for $k < 1.19968$ and unstable for $k > 1.19968$. They choose $k = 1/2$ and $k = 2$ as representative values. For small g, the neutral limiting mode has c equal to the velocity at the bottom for $\alpha = 0$, and $c = 0$ (*i.e.* the value of U at the inflection point) for $\alpha = \alpha_{\max}$.

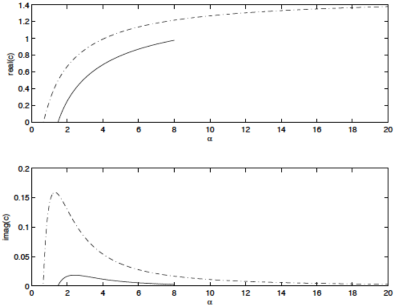

FIGURE 4. Unstable eigenvalues for cubic profile $U = y + y^3/2$. $g = 0$: -.-, $g = 1$: solid line.

The neutral limiting mode at $\alpha_{\min}$ has the inflection value $c = U(0) = 0$.

Result 1. Let $\Omega = R \times (-1, 1)$ and let the base flow profile be given by $U = -1$ for $-1 < y < 0$ and $U = 1$ for $0 < y < 1$. Then the stationary solution given by the base flow for the Euler equations with a rigid bottom and a free surface (given before) is linearly long wave stable if $0 < g < 1/2$ and linearly long wave unstable if $g = 0$ or $g > 1/2$.

Result 2. Let $\Omega = R \times (-L, 1)$ and $U = -1$ for $-L < y < 0$ and $U = 1$ for $0 < y < 1$. There exists a critical value g_c such that the stationary solution for the system (given before) is linearly long wave stable if $0 < g < g_c$ and linearly long wave unstable if $g = 0$ or $g > g_c$. We have $Lg_c \to 4$ as $L \to +\infty$ and $g_c \to 4$ as $L \to 0$.

More precisely $(L+1)g_c$ in terms of $\log L$ may be drawn.

Result 3. Let us consider a general shear flow with a base profile given by $U(y)$, which is smooth and strictly monotone in the sense that $U' \neq 0$. Then, in the limit of zero wave number, the linearization of system at the stationary solution does not have unstable eigenvalues if g is small enough.

Our result leaves open the possibility of instabilities for small wave number, with a wave speed whose imaginary part tends to zero as $\alpha \to 0$. However, such a situation can be expected to be nongeneric.

Result 4. There exists a non monotone flow profile U such that for small $g > 0$, system linearized around the stationary solution admits long wave instabilities.

Comments on A.G. VORONOVICH, E.D. LOBANOV, S.A. RYBAK's paper: Instability with piecewise linear profiles : vortex sheet at a fixed depth below the surface. Depth of bottom infinite in their situation: stable in their case.

The velocity profile in the top layer is linear, not constant. If we make it constant, i.e. if, in their notation, we set $d = 0$ and $u_0 = v_0$, then the conclusion changes. A double real eigenvalue at wave number 0, which splits up into a complex pair for positive wave number. That is, long waves are unstable for an infinite depth as might be expected from our second result.

Comments on N.A. Bakas and P.J. Ioannou: Piecewise linear profile in an half space.

Namely $\Omega = \Omega \times (-\infty, 0)$ with

$$\begin{aligned} U(y) &= V_0 \text{ if } -H_1 < y \leq 0, \\ &= V_0(y+H_2)/(H_2-H_1) \text{ if } -H_2 < y \leq -H_1, \\ &= 0 \text{ if } y \leq -H_2, \end{aligned}$$

- Small Froude number limit,
- Limit of no upper layer,
- Finite Froude number and upper layer depth

are considered.

More complete physical discussions:
Understanding and classification of the instabilities.

Studying interaction of the edge waves that arise at the density discontinuity at the surface and the vorticity waves that are supported at the mean vorticity gradient discontinuities in the interior.

Eigenvalue relation.

Due to the constant shear profile described in result 1 statement, (7) reduces to

$$-p'' + \alpha^2 p = 0 \tag{9}$$

in each layer, with continuity of p and $p'/(U-c)^2$ at the interface $y = 0$.

To get a solution, we obtain the eigenvalue relation

$$(-1+c^2)^2 \alpha \cosh(\alpha) + \alpha(-1+c)^4 \frac{\sinh^2(\alpha)}{\cosh(\alpha)} - 2(1+c^2) g \sinh(\alpha) = 0. \tag{10}$$

It is easy to discuss some limiting cases on (10):
1) $\alpha \to +\infty$, 2) $\alpha \to 0$ and 3) $g = 0$, $g \to +\infty$
with particular attention to long wave stability namely 2).

1) If $\alpha \to \infty$, we can replace the hyperbolic functions by exponentials. We then obtain the equation

$$2\alpha(-1+c)^2(1+c^2) - 2g(1+c^2) = 0, \tag{11}$$

which has the roots $\pm i$ and $1 \pm \sqrt{\dfrac{g}{\alpha}}$.
The former correspond to the Kelvin-Helmholtz instability at the interface,
The latter lead to Rayleigh-Taylor instability at the free surface if $g < 0$.

2) In the limit $\alpha = 0$, we find

$$c = \pm\sqrt{1 + g \pm \sqrt{4g + g^2}}. \tag{12}$$

In this case, all four roots are real if $0 \le g \le \dfrac{1}{2}$, otherwise there are complex roots.

3) In the case $g = 0$, we find a double root of 1 and the conjugate pair

$$c = -\frac{1}{\cosh(2\alpha)} \pm i \tanh(2\alpha). \tag{13}$$

In the opposite limit $g \to +\infty$, two roots converge to $\pm i$, while the other two behave like

$$\pm\sqrt{\frac{g\tanh(2\alpha)}{\alpha}}. \tag{14}$$

In conclusion in terms of Froude number:

If $\mathrm{Fr}^2 > 2$, long waves are stable, i.e. all eigenvalues c are real if α is small.

If Fr^2 smaller than 2, then this is no longer the case.

Two eigenvalues become a conjugate pair
and approach the "Kelvin-Helmholtz" eigenvalues as $\alpha \to \infty$.

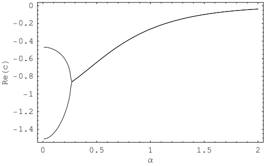

Figure: Real part of eigenvalues.

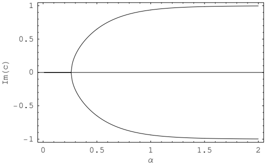

Figure: Imaginary part of eigenvalues.

The other two always remain real and approach the eigenvalues associated with the stable free surface as $\alpha \to \infty$.

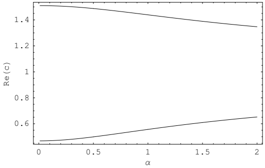

Figure: Real part of eigenvalues.

Effect of the size namely $\Omega = \mathbb{R} \times (-L, 1)$.

We place the bottom wall at $y = -L$: One condition replaced by $p'(-L) = 0$.
Eigenvalue relation quite complicated $\implies$ long wave limit $\alpha \to 0$.

In this case:

$$(1 - c^2)^2 - g((1 + c)^2 + L(1 - c)^2) = 0. \quad (15)$$

Always a root between $-\infty$ and -1 and another between 1 and ∞.
The remaining two roots, however, may become complex.

To delineate the boundary between real and complex roots, we determine the condition for (15) to have a double root. Using the Discriminant function of Mathematica, we found this to be the case when

$$-64 + 48g - 12g^2 + g^3 + 48gL + 84g^2L + 3g^3L - 12g^2L^2 + 3g^3L^2 + g^3L^3 = 0. \quad (16)$$

The function on the left of this equation has a sign change for $g > 0$ and positive derivative, so there is always a unique value of g where the equation is satisfied. In the limit $L \to 0$, we find

$$-64 + 48g - 12g^2 + g^3 = (g - 4)^3 = 0, \quad (17)$$

so the root is at $g = 4$.

To analyze the opposite limit $L \to \infty$, we set $g = k/L$. Collecting terms of the leading order L^3 in the resulting expression, we find

$$(k-4)^3 = 0. \tag{18}$$

That is, the limiting value of g behaves like $4/L$ as $L \to \infty$.

We computed the critical value of g as a function of L.

A minimum is found at $L = 1$, and at both extremes the limiting value is 4.

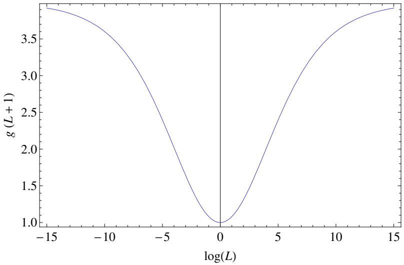

Figure: Critical value of $g(L+1)$.

Remark in terms of Froude number:

$L \to 0$: Internal interface is close to the bottom
$L \to +\infty$: Internal interface close to the free surface.

Long wave stability criterion for L close enough to 0: $\mathrm{Fr}^2 > 1/4$.
Long wave stability when L large: Fr has to be large enough.

Function $U(y)$ smooth and monotone: $U' > 0$.

In the long wave limit, $\alpha \to 0$, constant eigenfunction

$$p = 1$$

$\Longrightarrow$ next order to get the eigenvalue relation for $\alpha \to 0$.
Using HEISENBERG's methodology, expand p with respect to α

$$p = 1 + \alpha^2 q + \ldots$$

$\Longrightarrow$

$$(-\frac{q'}{(U-c)^2})' + \frac{1}{(U-c)^2} = 0, \tag{19}$$

and the boundary conditions are

$$q'(-1) = 0, \qquad \frac{q'(1)}{(U-c)^2} = \frac{1}{g}. \tag{20}$$

From this, we find

$$q'(y) = (U(y) - c)^2 \int_{-1}^{y} \frac{1}{(U(z) - c)^2} \, dz, \tag{21}$$

and the eigenvalue relation

$$\int_{-1}^{1} \frac{1}{(U - c)^2} \, dy = \frac{1}{g}. \tag{22}$$

Generalization for free surface of the Kelvin-Helmholtz instability criterion of Heisenberg for parallel plates which reads:

$$\int_{-1}^{1} \frac{1}{(U-c)^2}\, dy = 0$$

has non-real roots c with U the main profile.

Remark: Criterium for free surface in the paper written by J.C. Burns (1953).

We want to show that

$$\int_{-1}^{1} \frac{1}{(U-c)^2}\, dy = \frac{1}{g}$$

cannot have complex roots if g is positive and sufficiently small, *i.e.* if the right-hand side is positive and sufficiently large.

The integral on the left-hand side is real, then the real part of c must lie within the range of U. Moreover, if the value is to be large, then the imaginary part of c must be small.

Since $U(y)$ is assumed smooth and monotone, let $c = s + i\epsilon$, $U - s = v$, $h(v) = 1/U'(U^{-1}(s+v))$, $a = U(-1) - s$ and $b = U(1) - s$. With these notations, the integral on the left hand side of (22) transforms to

$$\int_a^b \frac{h(v)}{(v - i\epsilon)^2}\, dv. \tag{23}$$

We know that $a < 0$, $b > 0$ if we want (22) to be satisfied. Due to the assumed smoothness and monotonicity of U, h has a Taylor expansion in terms of v:

$$h(v) = h_0 + h_1 v + h_2 v^2 + \ldots \tag{24}$$

Inserting this expansion into the integral, we find

$$h_0\left(\frac{1}{a - i\epsilon} - \frac{1}{b - i\epsilon}\right) + h_1(\ln(b - i\epsilon) - \ln(a - i\epsilon)) + \ldots \tag{25}$$

where the dots indicate terms that remain bounded even as $\epsilon \to 0$. Hence the only case in which the integral can have a large value is if ϵ as well as either a or b is close to zero. Let us assume a is close to zero. Then we need

$$\frac{h_0}{a - i\epsilon} - h_1 \ln(a - i\epsilon) \tag{26}$$

to be large and positive. This, however, is not possible, since h_0 is positive and a is negative.

In this subsection, we prove that the result is not true if the velocity profile is not monotone.

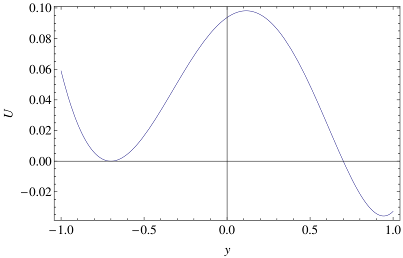

Figure: Nonmonotone velocity profile.

What matters for the following is that there are precisely two values a and b such that $U(a) = U(b) = 0$, where a is a velocity minimum: $U'(a) = 0$, $U''(a) > 0$, and we have $U'(b) < 0$, $U''(b) > 0$. We shall prove that there exist complex values c for which

$$\int_{-1}^{1} (U(y) - c)^{-2} \, dy \tag{27}$$

is arbitrarily large and positive.

Let N be a sufficiently small neighborhood of a; let I_1 be the integral over N, and let I_2 be the integral over $[-1,1]\backslash N$. We consider $c = s + i\epsilon$, where $s < 0$ and $\epsilon > 0$ are small in magnitude. From the discussion above, it is clear that

$$L = \lim_{s\to 0, \epsilon\to 0+} I_2 \tag{28}$$

exists and that the imaginary part of L is negative. Let us now consider I_1. We have

$$I_1 = \int_N \frac{1}{(U(y)-c)^2}\,dy = \int_N \frac{(U(y)-s)^2 - \epsilon^2 + 2(U(y)-s)i\epsilon}{((U(y)-s)^2+\epsilon^2)^2}\,dy. \tag{29}$$

We shall consider the range where $\epsilon << |s|$. In this case, the real and imaginary parts of I_1 are both positive. The real part becomes arbitrarily large if we let s approach zero. Now we fix s and vary ϵ. The imaginary part of I_1 is zero when $\epsilon = 0$, but becomes large if $\epsilon >> |s|^{5/2}$. There is a point in between where the imaginary part of I_1 cancels that of I_2, and hence $I_1 + I_2$ is real.

Local well-posedness

LOCAL WELL POSEDNESS WITH NO-IRROTATIONALITY CONDITION

Collaboration with M. RENARDY

Model and Theorem

The model (SW) in $\Omega = T^2$ or R^2

$$\begin{aligned} h_t + \operatorname{div}(h\mathbf{v}_1) &= 0, \\ -h_t + \operatorname{div}((1-h)\mathbf{v}_2) &= 0, \\ (\mathbf{v}_1)_t + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 + \frac{\rho-1}{\rho}\nabla h + \frac{1}{\rho}\nabla p &= \mathbf{0}, \\ (\mathbf{v}_2)_t + (\mathbf{v}_2\cdot\nabla)\mathbf{v}_2 + \nabla p &= \mathbf{0}. \end{aligned}$$

Remark. Indices 1 and 2 refer to the bottom and top layer respectively. Density of bottom layer $\rho = \rho_1/\rho_2 > 1$, the top one equals 1. The depth of the bottom layer is $h_1 = h$ and top $h_2 = 1-h$. Gravity g is taken equal to 1.

Theorem. Let $\rho > 1$ and $s > 2$. Assuming that $(h_0, \mathbf{v}_1^0, \mathbf{v}_2^0) \in (H^s)^5$ with $0 < h_0 < 1$ are such that

$$|\mathbf{v}_1^0 - \mathbf{v}_2^0|^2 < (\rho-1)(h_0 + \rho(1-h_0))/\rho. \qquad (30)$$

is satisfied and, moreover, $\operatorname{div}(h_0\mathbf{v}_1^0 + (1-h_0)\mathbf{v}_2^0) = 0$. Then, there exists $T_{\max} > 0$, and a unique maximal solution $(h, \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}([0, T_{\max}); (H^s)^5)$ (and a corresponding pressure p) to the system (SW), which satisfies the initial condition $(h, \mathbf{v}_1, \mathbf{v}_2)|_{t=0} = (h_0, \mathbf{v}_1^0, \mathbf{v}_2^0)$.

Framework and idea

Non-irrotational case: First result to the authors's knowledge.

Main result: Local well-posedness under optimal restrictions on the data by rewriting the system in an appropriate form which fits into the abstract theory of T.J.R. HUGHES, T. KATO and J.E. MARSDEN related to second order quasi-linear hyperbolic systems.

Idea: Isolate the "essential" part, using the total derivative $\partial_t + \mathbf{V} \cdot \nabla$ operator with $\mathbf{V}$ the weighted average velocity $\mathbf{V} = (\rho(1-h)\mathbf{v}_1 + h\mathbf{v}_2)/(\rho(1-h)+h)$.

Some Mathematical comments

Assumption equivalent to the one obtained by P. GUYENNE, D. LANNES, J.–C. SAUT [GLS2010] in the one-dimensional case (see $(24)_3$) and better than the one obtained in the irrotational case (see $(44)_3$). With our notation, Condition $(44)_3$ in [GLS2010] reads

$$\|\mathbf{v}_1^0 - \mathbf{v}_2^0\|_\infty^2 < (\rho - 1)(1 + \rho - (\rho - 1)\|2h_0 - 1\|_\infty)/2\rho.$$

We note we obtain if we replace the L^∞ norms by point values.

Methods in GLS2010:

In one-dimension, explicit relation between $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_2 = -h\mathbf{v}_1/(1-h)$. In the bi-fluid framework, no gravity inside, see B.L. KEYFITZ's works (reduction indicated due to C.M. DAFERMOS) related to singular shocks, Riemann problems and loss of hyperbolicity.

In irrotational-two dimensional case, non-local relation between $\mathbf{v}_1 = \nabla\Phi_1$ and $\mathbf{v}_2 = \nabla\Phi_2$ through

$$\mathrm{div}(h\nabla\Phi_1) = -\mathrm{div}((1-h)\nabla\Phi_2).$$

The interesting difficulty being to define an appropriate symmetrizer.

Some physical comments

Physical point of view: Condition arises from the competition between the Kelvin-Helmholtz instability and the stabilizing effect of gravity.

Same condition obtained in the study of long wave linear stability of density stratified two layer flow with a constant velocity in each layer (take the limit $k \to 0$ with surface tension coefficient $\gamma = 0$ and $g = 1$ in (3.6) of Funada-Joseph):

$$|\mathbf{v}_1 - \mathbf{v}_2|^2 \leq \left[\frac{\tanh(kh_1)}{\rho_1} + \frac{\tanh(kh_2)}{\rho_2}\right]\frac{1}{k}[(\rho_1 - \rho_2)g + \gamma k^2]$$

See also the recent fundamental mathematical paper D. LANNES in the nonlinear framework. Note that papers T. FUNADA – D.D. JOSEPH and D. LANNES concern potential flows.

Applications to bifluid systems and simulations.

Remark: Same kind of result in 3 dimension for $s > 5/2$ with application for the two-fluid models of a suspension page 903 (with no viscosity $\mu = 0$) in R. CAFLISH, G. PAPANICOLAOU (*SIAM J. Appl. Math* (1983)).

Remark: If no gravity and nothing more, well posedness only for analytical data (See E. GRENIER, *Comm. Partial Diff. Eqs* (1996)).

Remark: Important to deal with non-irrotational data in bifluid framework. For instance Bestion closure in the momentum equations.

$$P_{\text{int}} \nabla \alpha_i = \delta \frac{\alpha_1 \alpha_2 \rho_1 \rho_1}{\alpha_2 \rho_1 + \alpha_1 \rho_2} (u_1 - u_2)^2 \nabla \alpha_i$$

with $\delta \geq 1$.

Remark: Important from a numerical point of view: Iterative scheme!

Algebraic computations

We take the divergence of the last two equations and obtain

$$\begin{aligned}(\frac{\partial}{\partial t} + (\mathbf{v}_1 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_1 + \frac{\rho - 1}{\rho}\Delta h + \frac{1}{\rho}\Delta p &= -\,\mathrm{tr}\,((\nabla \mathbf{v}_1)^2),\\ (\frac{\partial}{\partial t} + (\mathbf{v}_2 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_2 + \Delta p &= -\,\mathrm{tr}\,((\nabla \mathbf{v}_2)^2).\end{aligned}$$

We can eliminate p and combine these two equations in the form

$$\begin{aligned}\rho(\frac{\partial}{\partial t} + (\mathbf{v}_1 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_1 &- (\frac{\partial}{\partial t} + (\mathbf{v}_2 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_2 + (\rho - 1)\Delta h\\ &= -\rho\,\mathrm{tr}\,((\nabla \mathbf{v}_1)^2) + \,\mathrm{tr}\,((\nabla \mathbf{v}_2)^2).\end{aligned}$$

We introduce the following weighted average of the velocity ("Favre velocity"):

$$V = \frac{\rho(1-h)\mathbf{v}_1 + h\mathbf{v}_2}{\rho(1-h) + h}.$$

Algebraic computations

With this, we can write in the form

$$
\begin{aligned}
&(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla))\,\mathrm{div}\,(\rho\mathbf{v}_1 - \mathbf{v}_2) + (\rho-1)\Delta h \\
&+\frac{\rho}{h+\rho(1-h)}(\mathbf{v}_1-\mathbf{v}_2)\cdot(h\nabla\,\mathrm{div}\,\mathbf{v}_1 + (1-h)\nabla\,\mathrm{div}\,\mathbf{v}_2) \\
= \quad & -\rho\,\mathrm{tr}\,((\nabla\mathbf{v}_1)^2) + \mathrm{tr}\,((\nabla\mathbf{v}_2)^2).
\end{aligned}
$$

Combining the first two equations, we find

$$\mathrm{div}\,(h\mathbf{v}_1 + (1-h)\mathbf{v}_2) = 0.$$

Using this, we find

$$
\begin{aligned}
&(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla))\,\mathrm{div}\,(\rho\mathbf{v}_1 - \mathbf{v}_2) + (\rho-1)\Delta h \\
&-\frac{\rho}{h+\rho(1-h)}((\mathbf{v}_1-\mathbf{v}_2)\cdot\nabla)^2 h = \mathbf{f}_1(\mathbf{v}_1,\mathbf{v}_2,h,\nabla\mathbf{v}_1,\nabla\mathbf{v}_2,\nabla h),
\end{aligned}
$$

where $\mathbf{f}_1$ depends only on the arguments indicated.

Algebraic computations

Next, we multiply the first equation of System ("mass equation") by $\rho(1-h)/(h+\rho(1-h))$, the second equation by $h/(h+\rho(1-h))$ and subtract. The result is

$$h_t + (\boldsymbol{V}\cdot\nabla)h + \frac{(1-h)h}{h+\rho(1-h)}\,\mathrm{div}\,(\rho\mathbf{v}_1 - \mathbf{v}_2) = 0.$$

We can now combine to find

$$(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla))^2 h = \frac{(1-h)h}{h+\rho(1-h)}\Big((\rho-1)\Delta h - \frac{\rho}{h+\rho(1-h)}((\mathbf{v}_1-\mathbf{v}_2)\cdot\nabla)^2 h\Big) + \mathbf{f}_2(\mathbf{v}_1,\mathbf{v}_2,h,\nabla\mathbf{v}_1,\nabla\mathbf{v}_2,\nabla h,h_t).$$

For given $\mathbf{v}_1$ and $\mathbf{v}_2$, this is a second order hyperbolic equation for h as long as

$$|\mathbf{v}_1-\mathbf{v}_2|^2 < (\rho-1)(h+\rho(1-h))/\rho.$$

Iterative scheme

Use an abstract result established by T.J.R. HUGHES, T. KATO and J.E. MARSDEN. We begin with a quote of the abstract theorem, see pages 275–276. This result concerns evolution problems of the form

$$\dot{u} = A(t,u)u + f(t,u),$$

where u takes values in a Banach space, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup, and f is a "perturbation" term. We say that $A \in G(X,M,\omega)$ if

$$\|e^{At}\|_{L(X)} \leq Me^{\omega t}.$$

The construction of the solution is by an iteration of the form

$$\dot{u}^{n+1} = A(t,u^n)u^{n+1} + f(t,u^n),$$

with fixed initial condition $u^n(0) = u_0$.

Hughes-Kato-Marsden Theorem

Theorem. Let $Y \subset Z \subset Z' \subset X$ be four real Banach spaces, all of them reflexive and separable, with continuous and dense inclusions. We assume that
1) Z' is an interpolation space between Y and X (i.e. linear operators which are bounded on both Y and X are also bounded on Z').
Let $N(X)$ be the set of all norms on X equivalent to the given one. On $N(X)$ we introduce a distance function

$$d(\|\cdot\|_\alpha, \|\cdot\|_\beta) := \ln \max\{\sup_{z\neq 0} \|z\|_\alpha/\|z\|_\beta, \sup_{z\neq 0} \|z\|_\beta/\|z\|_\alpha\}.$$

Let W be an open set in Y. We assume that there is a real number β and positive numbers $\lambda_N,\ \mu_N, \ldots$ such that the following hold for all $t, t' \in [0,T]$ and $w, w' \in W$.
2) $N(t,w) \in N(X)$, and

$$\begin{aligned} d(N(t,w), \|\cdot\|_X) &\leq \lambda_N, \\ d(N(t',w'), N(t,w)) &\leq \mu_N[|t'-t| + \|w'-w\|_Z]. \end{aligned}$$

Hughes-Kato-Marsden Theorem

3) There is an isomorphism $S(t,w) \in B(Y,X)$, with

$$\|S(t,w)\|_{Y,X} \leq \lambda_S, \; \|S(t,w)^{-1}\|_{X,Y} \leq \lambda_S',$$
$$\|S(t',w') - S(t,w)\|_{Y,X} \leq \mu_S[|t'-t| + \|w'-w\|_Z].$$

4) $A(t,w) \in G(X_{N(t,w)}, 1, \beta)$.
5) $S(t,w)A(t,w)S(t,w)^{-1} = A(t,w) + B(t,w)$, where $B(t,w)$ is a bounded operator in X and $\|B(t,w)\|_X \leq \lambda_B$.
6) $A(t,w) \in B(Y,Z)$ with

$$\|A(t,w)\|_{Y,Z} \leq \lambda_A, \; \|A(t,w') - A(t,w)\|_{Y,Z'} \leq \mu_A \|w'-w\|_{Z'}.$$

Moreover, the mapping $t \to A(t,w) \in B(Y,X)$ is continuous in norm.
7) $f(t,w) \in Y$, with

$$\|f(t,w)\|_Y \leq \lambda_f, \; \|f(t,w') - f(t,w)\|_{Z'} \leq \mu_f \|w'-w\|_{Z'},$$

and the mapping $t \to f(t,w) \in X$ is continuous.

Hughes-Kato-Marsden Theorem

If all of the above assumptions are satisfied, and $u_0 \in W \subset Y$, then there is a $T' \in (0, T]$ such that System (I) has a unique solution u on $[0, T']$ with $u \in C([0, T']; W) \cap C^1([0, T']; X)$. Here T' may depend on all the constants involved in the assumptions and on the distance between u_0 and the boundary of W. The mapping $u_0 \to u(t)$ is Lipschitz continuous in the Z'-norm, uniformly for $t \in [0, T']$. The solution is obtained by the iteration.

Application

To apply the result, we shall view $\mathbf{w}_i$ as determined by ω_i, ϕ_i, and $\mathbf{v}_i$ given by

$$\mathbf{v}_i = \mathbf{w}_i + \mathbf{q}_i + \nabla\phi_i.$$

Thus, $\mathbf{v}_i \in (H^s)^2$ is determined by $\omega_i \in H^{s-1}$, $\mathbf{q}_i \in \mathbb{R}^2$, $h \in H^s$, and $h_t \in H^{s-1}$. We set

$$u = (h, g, \omega_1, \omega_2, \mathbf{q}_1, \mathbf{q}_2),$$

where g represents h_t.
The spaces are given as

$$\begin{aligned} Y &= H^s \times H^{s-1} \times (H_0^{s-1})^2 \times \mathbb{R}^4, \\ Z &= Z' = H^{s-1} \times H^{s-2} \times (H_0^{s-2})^2 \times \mathbb{R}^4. \\ X &= H^1 \times L^2 \times (L_0^2)^2 \times \mathbb{R}^4. \end{aligned}$$

Here the subscript 0 denotes functions of zero average. We define W to be a sufficiently small neighborhood of the initial data in Y; in particular W must be small enough so that h and $1-h$ have strict lower bounds and (1) (with $\mathbf{v}_i$ given by through Helmotz decomposition) holds uniformly on W.

Application

We can set

$$S = ((-\Delta + 1)^{(s-1)/2})^4 \times (Id)^4.$$

Consider

$$u = (h, g, \omega_1, \omega_2, \mathbf{q}_1, \mathbf{q}_2), \qquad \tilde{u} = (\tilde{h}, \tilde{g}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2).$$

We define

$$A(\tilde{u})u = \begin{pmatrix} g, \\ -2(\tilde{V}\cdot\nabla)g - (\tilde{V}\cdot\nabla)^2 h + \dfrac{(1-\tilde{h})\tilde{h}}{\tilde{h}+\rho(1-\tilde{h})}\Big((\rho-1)\Delta h \\ -\dfrac{\rho}{\tilde{h}+\rho(1-\tilde{h})}((\tilde{\mathbf{v}}_1-\tilde{\mathbf{v}}_2)\cdot\nabla)^2 h\Big), \\ -(\tilde{\mathbf{v}}_1\cdot\nabla)\omega_1 - \omega_1 \operatorname{div}\tilde{\mathbf{v}}_1, \\ -(\tilde{\mathbf{v}}_2\cdot\nabla)\omega_2 - \omega_2 \operatorname{div}\tilde{\mathbf{v}}_2, \\ \mathbf{0}, \\ \mathbf{0} \end{pmatrix}$$

where $\tilde{\phi}_i$, $\tilde{\mathbf{v}}_i$ and $\tilde{V}$ are given in terms of $\tilde{u}$ through the relations (mass equation, algebraic relation...).

Application

Moreover, we define

$$\begin{aligned}(N(\widetilde{u})u)^2 &= \|h\|^2 + \|\Big(\frac{(1-\tilde{h})\tilde{h}}{\tilde{h}+\rho(1-\tilde{h})}\Big)^{1/2}(\rho-1)^{1/2}\nabla h\|^2 \\ &- \|\frac{(\rho(1-\tilde{h})\tilde{h})^{1/2}}{\tilde{h}+\rho(1-\tilde{h})}((\tilde{\mathbf{v}}_1-\tilde{\mathbf{v}}_2)\cdot\nabla)h\|^2 + \|g+(\tilde{V}\cdot\nabla)h\|^2 \\ &+ \|\omega_1\|^2 + \|\omega_2\|^2 + |\mathbf{q}_1|^2 + |\mathbf{q}_2|^2.\end{aligned}$$

The verification of the assumptions is quite routine using the definition of W, $N(w)$, S and $A(w)$. Assumption 4 follows from the Lumer-Phillips theorem (dissipativity of $A(w)$ and surjectivity of $A(w)-\lambda_0\mathrm{Id}$ for some $\lambda_0>0$ with the appropriate constants and norm), and the remaining assumptions can be verified using the fact that H^{s-1} is a Banach algebra, as well as a multiplier in lower order Sobolev spaces. The reader is referred to paper by HUGHES-KATO-MARSDEN for an application to nonlinear elasto-dynamics of the Theorem for which a similar system is involved.

Remarks

In D.B., B. DESJARDINS, J.–M. GHIDAGLIA, E. GRENIER. Low Mach Number Limit and Bi-Fluid systems. In preparation (2011).

1) Appropriate unknowns imply adequate variables to study low Mach number limits. System closed to the non-isentropic system.

2) With bestion term, the incompressible bi-fluid system gives

$$(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla))^2\alpha_+ = \frac{(1-\alpha_+)\alpha_+}{\alpha_+ + \rho(1-\alpha_+)}\Big(\frac{\delta\rho}{\alpha_+ + (1-\alpha_+)\rho}|u_+ - u_-|^2\Delta\alpha_+ - \frac{\rho}{\alpha_+ + \rho(1-\alpha_+)}((u_+ - u_-)\cdot\nabla)^2\alpha_+\Big) + \mathbf{f}_2(u_+, u_-, \alpha_+, \nabla u_+, \nabla u_-, \nabla\alpha_+, (\alpha_+)_t)$$

with $\mathbf{f}_2(u_+, u_+, \alpha_+, \nabla u_+, \nabla u_-, \nabla\alpha_+, (\alpha_+)_t) = 0$