Entropy MUSCL scheme by involving a MOOD approach

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Introduction

- Hyperbolic system of conservation laws

$$\begin{cases} \partial_t w + \partial_x f(w) = 0 \\ w(x,0) = w_0(x) \end{cases}$$

  $w : \mathbb{R}^+ \times \mathbb{R} \to \Omega$: unknown state vector
  $f : \Omega \to \mathbb{R}^d$: continuous flux function
  $w_0 \in L^1_{\text{loc}}(\mathbb{R};\Omega)$: initial condition
- $\Omega \subset \mathbb{R}^d$ convex set of physical states
- A convex function $S \in C^2(\Omega;\mathbb{R})$ is an entropy for the system if there exists an entropy flux $g \in C^2(\Omega;\mathbb{R})$ such that $\nabla f(w)\nabla S(w) = \nabla g(w), \forall w \in \Omega$
- Objective: study the stability of high-order space-time schemes

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t E + \partial_x (E + p) u = 0 \end{cases}$$

ρ density
u velocity
E total energy
p pressure given by the perfect gas law

$$p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right), \quad \gamma \in (1, 3]$$

- Set of physical states:

$$\Omega = \left\{ w \in \mathbb{R}^3,\ \rho > 0,\ p > 0 \right\}$$

- Entropy inqualities

$$\partial_t \rho \mathcal{G}(s) + \partial_x \rho \mathcal{G}(s) u \leq 0 \qquad s = \ln\left(\frac{p}{\rho^\gamma}\right)$$

A general high-order space-time scheme

Initial condition

$$w_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w_0(x)\,dx$$

Runge-Kutta time discretization

$$w_i^{n,(\ell)} = w_i^n - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right), \quad \ell = 1, \cdots, m$$

$$w_i^{n,(0)} = w_i^n, \quad w_i^{n+1} = w_i^{n,(m)}$$

Assumptions: $c_{\ell,j} \geq 0, \quad \sum_{j=0}^{m-1} c_{m,j} = 1$

Space discretization

$$F_{i+1/2}^{n,(j)} = F\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)} \right)$$

Assumptions: F continuous and consistent $(F(w, \cdots, w) = f(w))$

A general high-order space-time scheme

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$$w_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w_0(x)\, dx$$

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$$w_i^{n,(\ell)} = w_i^n - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right), \quad \ell = 1, \cdots, m$$

$$w_i^{n,(0)} = w_i^n, \quad w_i^{n+1} = w_i^{n,(m)}$$

We introduce the piecewise constant functions

$$w^{\Delta}(x,t) = w_i^n, \quad \text{for } (x,t) \in R_i^n,$$

$$w^{\Delta,(\ell)}(x,t) = w_i^{n,(\ell)}, \quad \text{for } (x,t) \in R_i^n.$$

Lax-Wendroff Theorem

Theorem

(i) *Assume the following hypotheses:*

- *There exists a compact* $K \subset \Omega$ *such that* $w^{\Delta,(\ell)} \in K$, $\forall \ell = 0, \cdots, m$;
- w^{Δ} *converges in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ *to a function* w.

Then w *is a weak solution.*

(ii) *Assume the additional hypothesis:*

- *For all entropy pair* (S, g), *there exists an entropy numerical flux* G, *such that we have the discrete entropy inequality (DEI)*

$$\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,j}}{\Delta x} \leq 0,$$

with $G_{i+1/2}^{n,(j)} = G\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)}\right)$.

Then w *is an entropic solution.*

High-order time schemes

Reformulation of the Runge-Kutta discretization (Shu-Osher)

$$w_i^{n,\ell} = \sum_{j=0}^{\ell-1} \left(\alpha_{\ell,j} w_i^{n,(j)} - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right) \right),$$

$$\text{with } \alpha_{\ell,j} > 0,\ \textstyle\sum_{j=0}^{\ell-1} \alpha_{\ell,j} = 1.$$

Assumption: $\beta_{\ell,j} > 0$.

Theorem

Assume the first-order time scheme $w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$ *satisfies the DEI* $\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0$*, then the Runge-Kutta scheme satisfies the DEI*

$$\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}}{\Delta x} \leq 0$$

High-order space schemes

- No DEI like

$$\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}}{\Delta x} \leq 0$$

  was ever proven as soon as the scheme is at least second-order in space.

- Example: second-order MUSCL scheme
  - We consider $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ a slope limiter and we define the limited slope $\mu_i^{n,(j)} = L\left(w_i^{n,(j)} - w_{i-1}^{n,(j)}, w_{i+1}^{n,(j)} - w_i^{n,(j)}\right)$.
  - The MUSCL flux is defined by

$$F_{i+1/2}^{n,(j)} = F\left(w_i^{n,(j)} + \frac{1}{2}\mu_i^{n,j}, w_{i+1}^{n,(j)} - \frac{1}{2}\mu_{i+1}^{n,(j)}\right),$$

  where F is a first-order numerical flux.

MUSCL entropy inequalities : First-order assumptions

- Piecewise constant approximations:

$$w^h(x, t^n) = w_i^n \qquad x \in (x_{i-1/2}, x_{i+1/2})$$

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) \quad \text{with} \quad F_{i+1/2}^n = F(w_{i+1}^n, w_i^n)$$

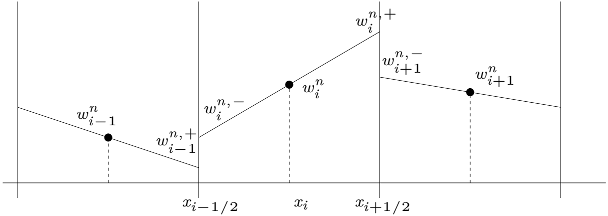
CFL restriction: $\dfrac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left(u \pm \sqrt{\dfrac{\gamma p}{\rho}} \right)_{i+1/2}^n \right) \leq \dfrac{1}{2}$

- Ω-preserving: if $w_i^n \in \Omega$ then $w_i^{n+1} \in \Omega$
- Discrete entropy inequalities

$$\rho_i^{n+1} \mathcal{G}(\ln s_i^{n+1}) - \rho_i^n \mathcal{G}(\ln s_i^n) + \frac{\Delta t}{\Delta x} \left(\{\rho \mathcal{G}(\ln s) u\} (w_{i+1}^n, w_i^n) - \{\rho \mathcal{G}(\ln s) u\} (w_i^n, w_{i-1}^n) \right) \leq 0$$

Second-order MUSCL schemes derivation

- Piecewise constant approximations



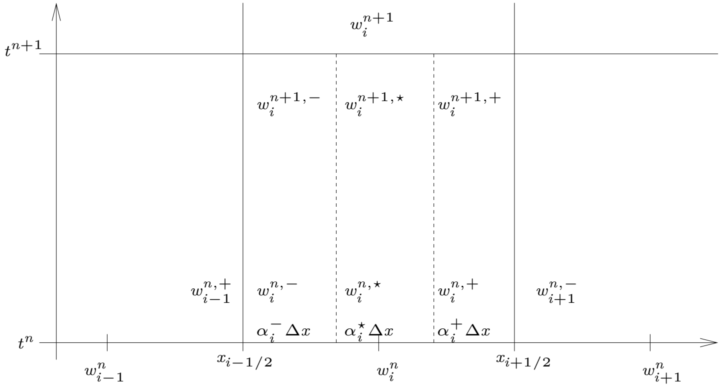
- MUSCL : van Leer(79)

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F^{\pm}_{i+1/2} - F^{\pm}_{i-1/2}) \qquad F^{\pm}_{i+1/2} = F(w_{i+1}^{n,-}, w_i^{n,+})$$

Inner approximations $w_i^{n,\pm} = w^h(x_{i\pm 1/2}, t^n) = w_i^n + \Delta w_i^{n,\pm}$
Here, $\Delta w_i^{n,\pm}$ given by a limitation procedure L
But not necessarily in conservation form $\frac{1}{2}\left(w_i^{n,-} + w_i^{n,+}\right) \neq w_i^n$

Scheme reformulation (Perthame-Shu 95)

- Let us introduce $\alpha_i^- \geq 0$ $\alpha_i^\star \geq 0$ $\alpha_i^+ \geq 0$ such that $\alpha_i^- + \alpha_i^\star + \alpha_i^+ = 1$
- We define $w_i^{n,\star}$ as follows $\alpha_i^- w_i^{n,-} + \alpha_i^\star w_i^{n,\star} + \alpha_i^+ w_i^{n,+} = w_i^n$



- Evolution of the inner intermediate states by considering the associated first-order scheme

$$w_i^{n+1,-} = w_i^{n,-} - \frac{\Delta t}{\alpha_i^- \Delta x}\Big(F(w_i^{n,-}, w_i^{n,\star}) - F(w_{i-1}^{n,+}, w_i^{n,-})\Big)$$

$$w_i^{n+1,\star} = w_i^{n,\star} - \frac{\Delta t}{\alpha_i^\star \Delta x}\Big(F(w_i^{n,\star}, w_i^{n,+}) - F(w_i^{n,-}, w_i^{n,\star})\Big)$$

$$w_i^{n+1,+} = w_i^{n,+} - \frac{\Delta t}{\alpha_i^+ \Delta x}\Big(F(w_i^{n,+}, w_{i+1}^{n,-}) - F(w_i^{n,\star}, w_i^{n,+})\Big)$$

- CFL condition: sub-grid CFL

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\frac{1}{\alpha_i^{\pm\star}} \left(u \pm \sqrt{\frac{\gamma p}{\rho}} \right)_{i+1/2}^{n,\pm\star} \right) \leq \frac{1}{2}$$

Reformulation of the MUSCL scheme

$$w_i^{n+1} = \alpha_i^- w_i^{n+1,-} + \alpha_i^\star w_i^{n+1,\star} + \alpha_i^+ w_i^{n+1,+}$$

Stability and robustness

Theorem

Assume $w_i^{n,\pm\star} \in \Omega$ *then*

1. $w_i^{n+1} \in \Omega$
2. *Discrete entropy inequalities*

$$\rho_i^{n+1}\mathcal{G}(s_i^{n+1}) - \overline{\rho\mathcal{G}(s)}_i^n + \frac{\Delta t}{\Delta x}\Big(\{\rho\mathcal{G}(s)u\}(w_i^{n,+}, w_{i+1}^{n,-}) - \{\rho\mathcal{G}(s)u\}(w_{i-1}^{n,+}, w_i^{n,-})\Big) \leq 0$$

$$\overline{\rho\mathcal{G}(s)}_i^n = \alpha_i^- \rho_i^{n,-}\mathcal{G}(s_i^{n,-}) + \alpha_i^\star \rho_i^{n,\star}\mathcal{G}(s_i^{n,\star}) + \alpha_i^+ \rho_i^{n,+}\mathcal{G}(s_i^{n,+})$$

Remark : If $\frac{1}{2}\left(w_i^{n,-} + w_i^{n,+}\right) = w_i^n$ then

$$\alpha^\star = 0 \qquad \text{and} \qquad \alpha^+ = \alpha^- = \frac{1}{2}$$

DEI satisfied by the MUSCL scheme

The known DEI satisfied by the MUSCL scheme all write

$$\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}}{\Delta x} \leq \sum_{j=0}^{m-1} \alpha_{m,j} \frac{P_i^{n,(j)} - S(w_i^{n,(j)})}{\Delta t}$$

where $P_i^{n,(j)} = P\left(w_i^{n,(j)}, \mu_i^{n,(j)}, \Delta x, S\right)$.

- Examples of operator P:

$$P_1(w, \mu, \Delta x, S) = \frac{S(w - \mu/2) + S(w + \mu/2)}{2}$$

$$P_2(w, \mu, \Delta x, S) = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} S\left(w + \frac{x}{\Delta x}\mu\right) dx \quad \text{(Bouchut)}$$

- The operator P satisfies: $\exists C > 0$ such that

$$0 \leq P(w, \mu, \Delta x, S) - S(w) \leq C \|\nabla^2 S(w)\| \|\mu\|^2$$

Convergence of D^{Δ}: theoretical study

- We define the piecewise constant function

$$D^{\Delta}(x,t) = \sum_{j=0}^{m-1} \alpha_{m,j} \frac{P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)}{\Delta t}, \quad \text{for } (x,t) \in R_i^n$$

- Let μ be the weak-star limit of the sequence D^{Δ}. Let β be the entropy dissipation measure defined as the weak-star limit of the sequence

$$b^{\Delta}(x,t) = \sum_{j=0}^{m-1} \alpha_{m,j} \frac{\left\| w_i^{n,(j)} - w_{i-1}^{n,(j)} \right\|^2}{\Delta x}, \quad \text{for } (x,t) \in R_i^n.$$

- μ is absolutely continuous with respect to β.

Conjecture (Hou-le Floch)

The entropy dissipation measure β is concentrated on the curves of discontinuity of w.

Numerical study: test cases

1–rarefaction

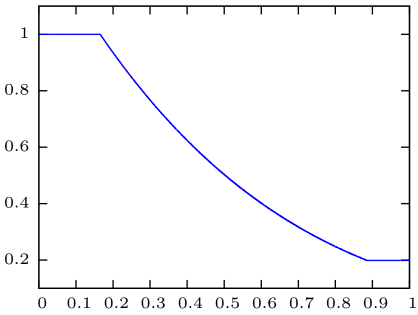


Figure : Exact solution in density

shock–shock

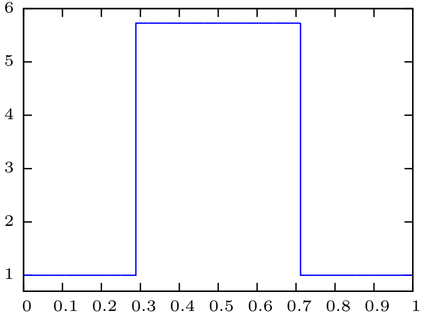


Figure : Exact solution in density

- L^1 error for the convergence: $E^\Delta = \sum_i \left|\rho_i^N - \rho_{ex}(x_i, T)\right|$
- Convergence of D^Δ: $I^\Delta = \int_{[0,1]\times[0,T]} D^\Delta(x,t)\,dxdt$

1–rarefaction: convergence of first-order time schemes

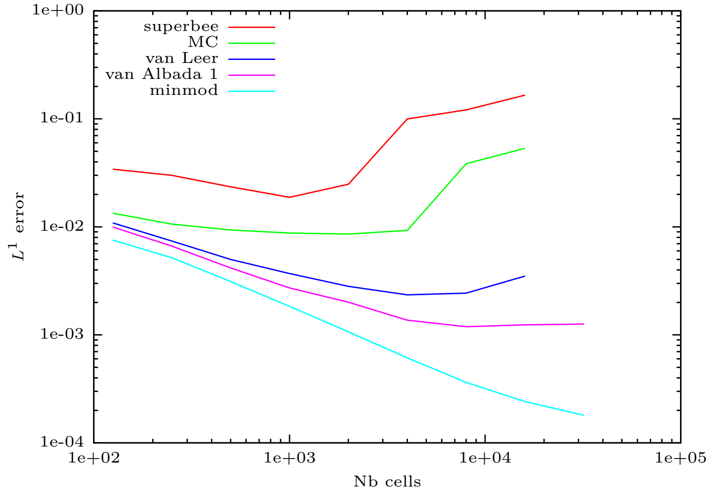


Figure : Convergence of second-order space / first-order time schemes

1–rarefaction: what superbee does (1)

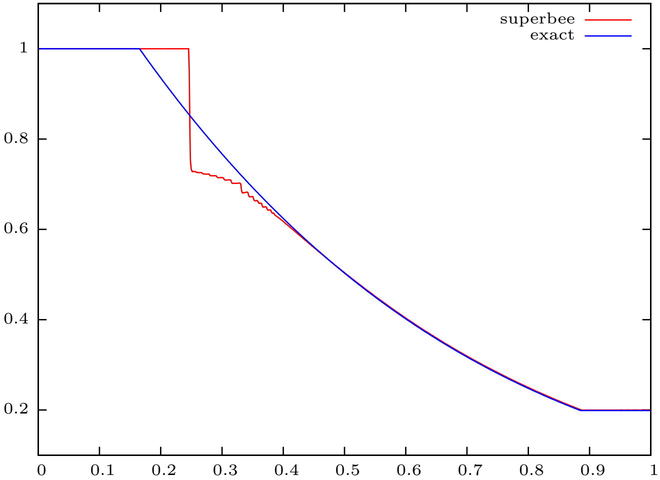


Figure : Solution given by the superbee limiter with 1000 cells

1–rarefaction: what superbee does (2)

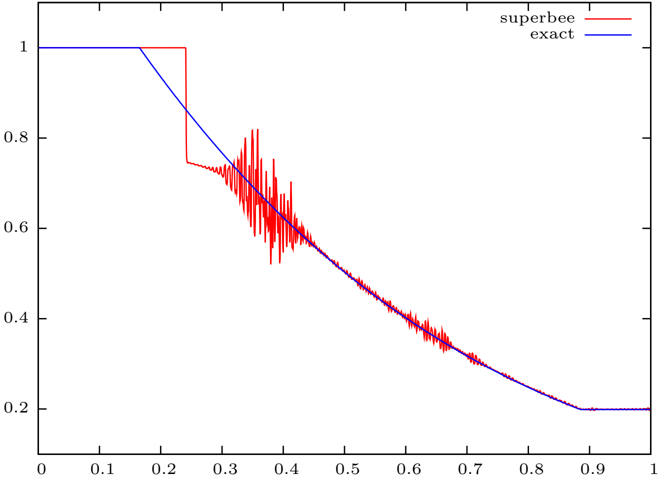


Figure : Solution given by the superbee limiter with 2000 cells

1–rarefaction: convergence of I^{Δ} for first-order time schemes

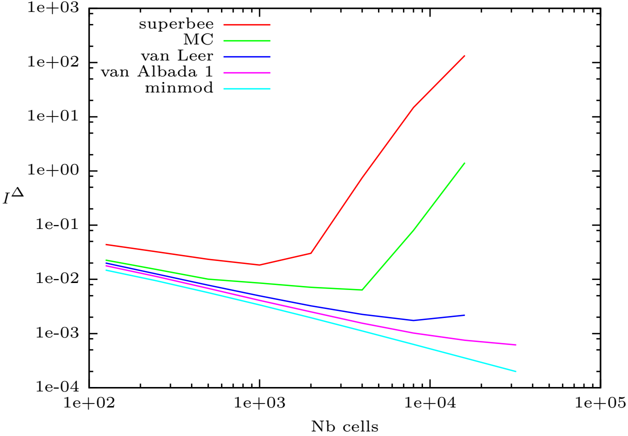


Figure : Convergence of I^{Δ} for second-order space / first-order time schemes

1–rarefaction: convergence of second-order time schemes

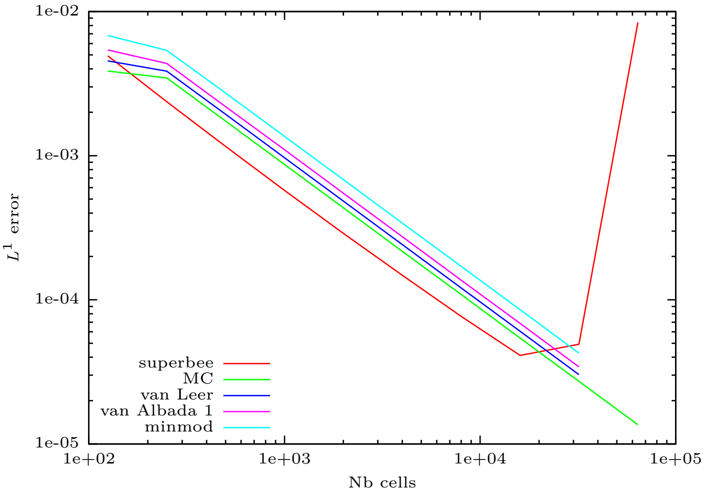


Figure : Convergence of second-order space-time schemes

1–rarefaction: convergence of I^{Δ} for second-order time schemes

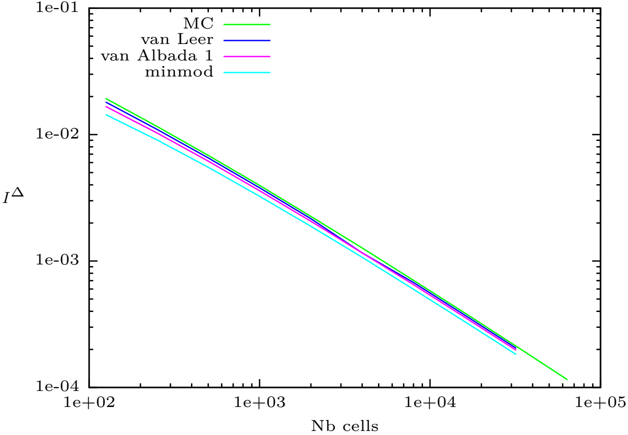


Figure : Convergence of I^{Δ} for second-order space-time schemes

Shock-Shock: convergence of first-order time schemes

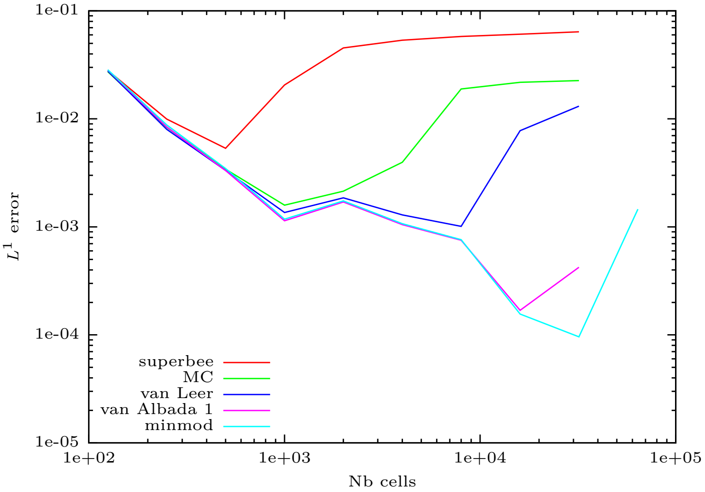


Figure : Convergence of second-order space / first-order time schemes

Shock-Shock: convergence of second-order time schemes

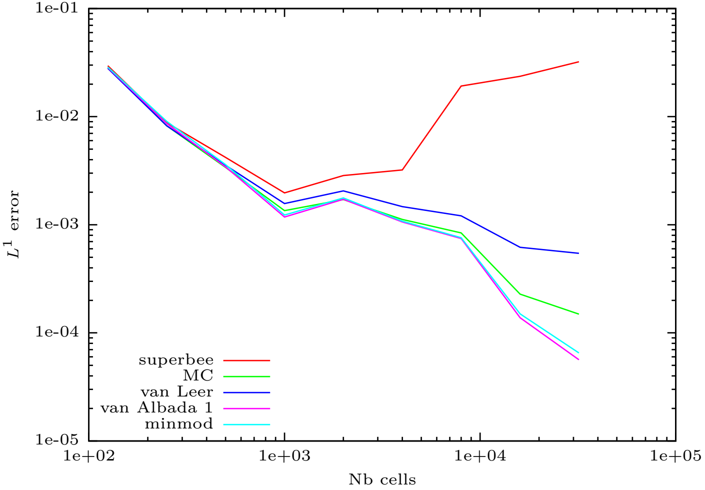


Figure : Convergence of second-order space-time schemes

Shock-Shock: convergence of I^{Δ} for second-order time schemes

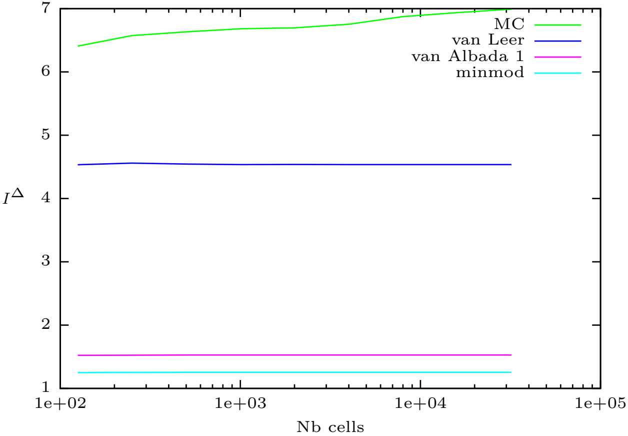


Figure : Convergence of I^{Δ} for second-order space-time scheme

Conclusion

- Numerical results confirm the Hou-le Floch conjecture: when the scheme converges, the weak-star limit μ of D^{Δ} seems to be concentrated on the curves of discontinuity of w.
- This does not imply that the limit is not entropic, but only that the usual DEI are not the suitable tool to prove a Lax-Wendroff theorem.
- We have to focus on the stronger DEI

$$\frac{S(w_i^{n+1}) - S(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,j}}{\Delta x} \leq 0,$$

- Most of the limiters, when combined with a first-order scheme, seem to be unstable to small perturbations, though with a very low explosion rate.

Objective: A posteriori entropy estimations
Unrealistic with an infinity number of entropy inqualities

The family of entropies for the Euler equations

The Euler system possesses a family of entropy pairs (S, g) written

$$S = -\rho \mathcal{G}(s), \quad g = -\rho u \mathcal{G}(s),$$

where $s = \ln\left(\frac{p}{\rho^\gamma}\right)$ is the specific entropy and $\mathcal{G}$ is a smooth function satisfying

$$\mathcal{G}'(s) > 0, \quad \mathcal{G}'(s) - \gamma \mathcal{G}''(s) > 0.$$

Lemma (reformulation)

The entropy pairs of the Euler system write

$$S(r) = \rho\psi(r), \quad g(r) = \rho u \psi(r),$$

where $r = \frac{\rho^{1/\gamma}}{p}$ and ψ is a smooth decreasing convex function.

We consider the scheme $w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right)$, where $w_i^n = (\rho_i^n, \rho_i^n u_i^n, E_i^n)^T$ and $F_{i+1/2} = (F_{i+1/2}^{\rho}, F_{i+1/2}^{\rho u}, F_{i+1/2}^{E})^T$.

Theorem

Assume the scheme is Ω-preserving. Assume the DEI

$$-\rho_i^{n+1} r_i^{n+1} \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(-F_{i+1/2}^{\rho} r_{i+1/2}^n + F_{i-1/2}^{\rho} r_{i-1/2}^n\right)$$

with $r_{i+1/2}^n = \begin{cases} r_{i+1}^n & \text{if} \quad F_{i+1/2}^{\rho} < 0 \\ r_i^n & \text{if} \quad F_{i+1/2}^{\rho} > 0 \end{cases}$. *Assume the additional CFL like condition (Larrouturou)*

$$\frac{\Delta t}{\Delta x}\left(\max(0, F_{i+1/2}^{\rho}) - \min(0, F_{i-1/2}^{\rho})\right) \leq \rho_i^n.$$

Then the scheme satisfies all the discrete entropy inequalities.

Example : the HLLC/Suliciu relaxation scheme

Proof of the Theorem (1)

The numerical flux can be written

$$F^{\rho}_{i+1/2} r_{i+1/2} = F^{\rho}_{i+1/2} \frac{r_i^n + r_{i+1}^n}{2} - \left| F^{\rho}_{i+1/2} \right| \frac{r_{i+1}^n - r_i^n}{2}.$$

The DEI then writes

$$r_i^{n+1} \geq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n,$$

where we have set

$$a = \frac{\Delta t}{2\Delta x} \left(F^{\rho}_{i-1/2} + \left| F^{\rho}_{i-1/2} \right| \right),$$

$$b = \rho_i^n - \frac{\Delta t}{2\Delta x} \left(F^{\rho}_{i+1/2} + \left| F^{\rho}_{i+1/2} \right| - F^{\rho}_{i-1/2} + \left| F^{\rho}_{i-1/2} \right| \right),$$

$$c = \frac{\Delta t}{2\Delta x} \left(\left| F^{\rho}_{i+1/2} \right| - F^{\rho}_{i+1/2} \right).$$

Proof of the Theorem (2)

- - We have $a + b + c = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^{\rho}_{i+1/2} - F^{\rho}_{i-1/2}\right) = \rho_i^{n+1}$.
  - $a \geq 0, \; c \geq 0$
  - $b \geq 0$ thanks to the CFL like condition

  $\Rightarrow r_i^{n+1}$ is greater than a convex combination of r_{i-1}^n, r_i^n and r_{i+1}^n.
- We consider an entropy pair which can writes $(S, g) = (\rho\psi(r), \rho u \psi(r))$ with ψ a smooth decreasing convex function thanks to the Lemma.
- ψ is decreasing:

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n\right)$$

- Jensen inequality (ψ is convex):

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}}\psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}}\psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}}\psi\left(r_{i+1}^n\right)$$

Proof of the Theorem (3)

- We replace a, b and c by their value to obtain

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) - \frac{\Delta t}{2\Delta x}\Big(F^\rho_{i+1/2}(\psi(r_i^n)+\psi(r_{i+1}^n)) - |F^\rho_{i+1/2}|(\psi(r_{i+1}^n)-\psi(r_i^n)) - F^\rho_{i-1/2}(\psi(r_{i-1}^n)+\psi(r_i^n)) + |F^\rho_{i-1/2}|(\psi(r_i^n)-\psi(r_{i-1}^n))\Big).$$

- We define $\psi^n_{i+1/2} = \begin{cases} \psi\left(r^n_{i+1}\right) & \text{if} \quad F^\rho_{i+1/2} < 0 \\ \psi\left(r^n_i\right) & \text{if} \quad F^\rho_{i+1/2} > 0 \end{cases}$.
- We have shown the DEI (for the entropy pair $(\rho\psi(r), \rho u\psi(r))$)

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2}\psi^n_{i+1/2} - F^\rho_{i-1/2}\psi_{i-1/2}\right).$$

□

First-order scheme

We consider a first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(F\left(w_i^n, w_{i+1}^n\right) - F\left(w_{i-1}^n, w_i^n\right)\right).$$

For a time step restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left|\lambda^{\pm}\left(w_i^n, w_{i+1}^n\right)\right| \leq \frac{1}{2},$$

the first-order scheme is assumed to satisfy:

(i) $w_i^n \in \Omega, \quad \forall i \in \mathbb{Z} \quad \Rightarrow \quad w_i^{n+1} \in \Omega, \quad \forall i \in \mathbb{Z}$

(ii) $\forall i \in \mathbb{Z}$, the following DEI is satisfied:

$$-\rho_i^{n+1} r_i^{n+1} \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x} \Big(-F^{\rho}\left(w_i^n, w_{i+1}^n\right) r_{i+1/2}^n + F^{\rho}\left(w_{i-1}^n, w_i^n\right) r_{i-1/2}^n \Big).$$

Example: HLLC scheme

High-order reconstruction

- A reconstruction function is a continuous function $\mathcal{R} : \Omega^{2s+1} \to \Omega$ such that $\mathcal{R}(w, \cdots, w) = w$, for all $w \in \Omega$.
- A high-order reconstruction function is usually a reconstruction function based on high degree polynomial reconstruction.
- Here, we consider two reconstruction functions $\mathcal{R}_-$ and $\mathcal{R}_+$. The associated MUSCL scheme is then given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(F\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right) - F\left(\mathcal{W}_{i-1,+}, \mathcal{W}_{i,-}\right)\right),$$

with $\mathcal{W}_{i,\pm} = \mathcal{R}_\pm \left(w_{i-s}^n, \cdots, w_{i+1}^n\right)$

- Example: second-order MUSCL scheme:

$$\mathcal{R}_\pm \left(w_{i-1}^n, w_i^n, w_{i+1}^n\right) = w_i^n \pm \frac{1}{2} L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n\right),$$

where L is a slope limiter.

The E-MOOD algorithm

1. **Evaluation of the reconstructed states.** The reconstructed states are given by $\mathcal{W}_{i,\pm} = \mathcal{R}_{\pm}\left(w_{i-s}^n, \cdots, w_{i+s}^n\right)$
2. **Computation of the candidate solution w_i^*.** We compute a candidate solution w_i^* using the MUSCL scheme

$$w_i^* = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right) - F\left(\mathcal{W}_{i-1,+}, \mathcal{W}_{i,-}\right)\right).$$

3. **DEI test.** If w_i^* does not satisfy the DEI test

$$-\rho_i^* r_i^* \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(-F_{i+1/2}^{\rho} r_{i+1/2}^n + F_{i-1/2}^{\rho} r_{i-1/2}^n\right),$$

with $F_{i+1/2}^{\rho} = F^{\rho}\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right)$, then we set $\mathcal{W}_{i,\pm} = w_i^n$

4. **Stopping criterion.**
   - If the DEI test is satisfied on all the cells, the candidate solution is valid and we set $w_i^{n+1} = w_i^*$
   - else the solution is recomputed from step 2

Stability and robustness of the E-MOOD scheme

Theorem

Assume the time step Δt is chosen in order to satisfy the two following CFL like conditions:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| \lambda^{\pm} \left(w_{i,+}, w_{i+1,-} \right) \right|, \left| \lambda^{\pm} \left(w_{i,-}, w_{i,+} \right) \right| \right) \leq \frac{1}{4}$$

$$\frac{\Delta t}{\Delta x} \left(\max(0, F^{\rho}_{i+1/2}) - \min(0, F^{\rho}_{i-1/2}) \right) \leq \rho_i^n .$$

Then the E-MOOD method provides an updated solution w_i^{n+1} after a finite number of iterations. It is physically admissible, and it satisfies all the entropy inequalities.

1–rarefaction: first-order time schemes

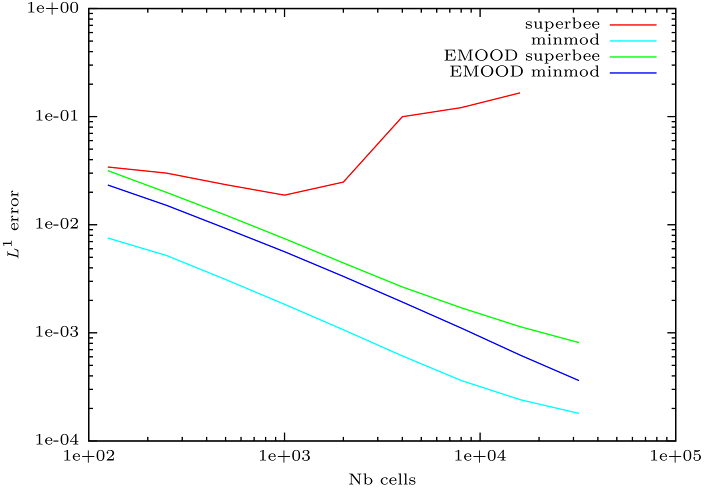


Figure : Convergence of first-order time schemes: E-MOOD vs MUSCL

1–rarefaction: second-order time schemes

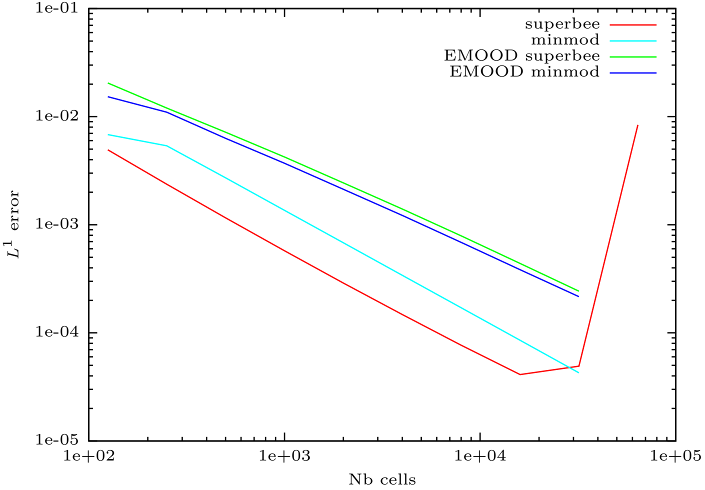


Figure : Convergence of second-order time schemes: E-MOOD vs MUSCL

Shock-Shock: first-order time schemes

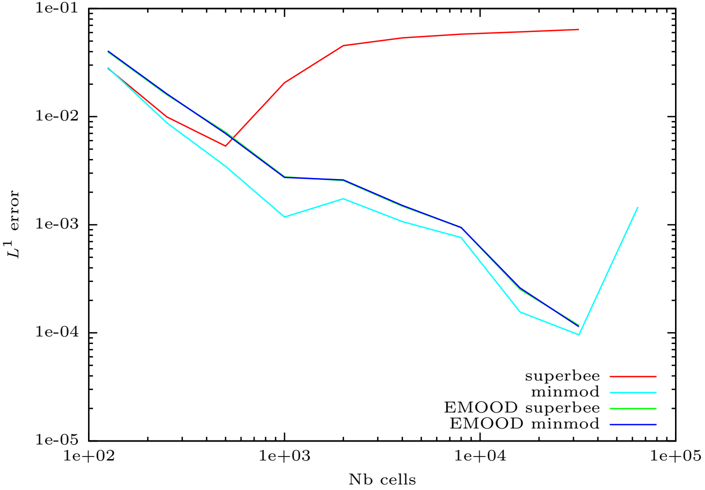


Figure : Convergence of first-order time schemes: E-MOOD vs MUSCL

Shock-Shock: second-order time schemes

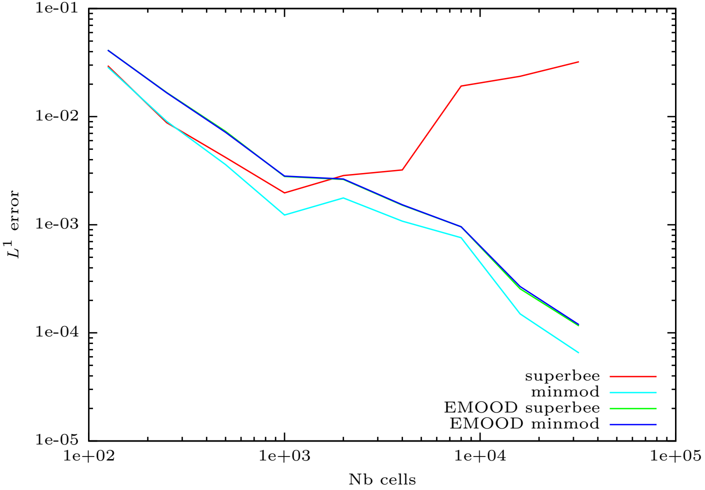


Figure : Convergence of second-order time schemes: E-MOOD vs MUSCL

Smooth problem

- $$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.2 \text{ or } x > 0.8 \\ 1 + \exp\left(\frac{(x-0.5)^2}{(x-0.2)(x-0.8)}\right) & \text{if } 0.2 \leq x \leq 0.8 \end{cases}$$
  $u_0(x) = 1,\ p_0(x) = 1$
- Periodic boundary conditions

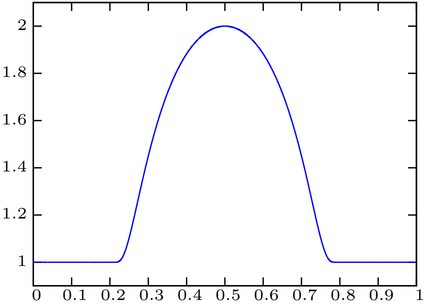


Figure : Initial and final solution in density for the smooth problem

Smooth problem: convergence

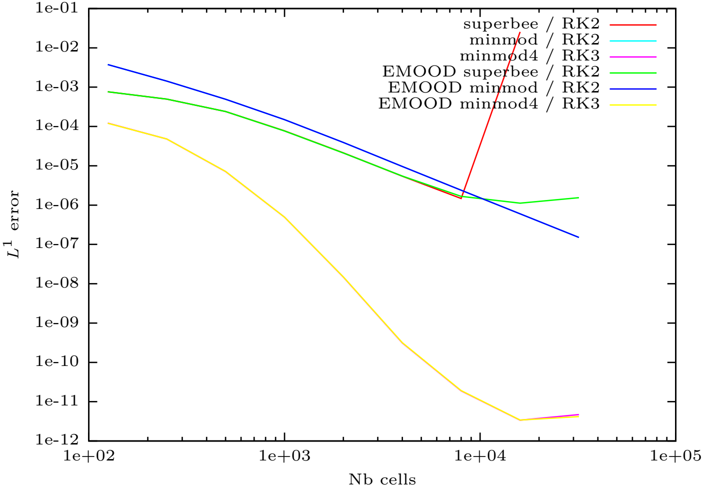


Figure : Convergence: E-MOOD vs MUSCL

Thank you for your attention