On some numerical schemes for fluid mechanics

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Models in fluids mechanics

Euler Equations, Incompressible and compressible Navier-Stokes Equations, Multiphase Flows...

Example : Euler $d = 2$ or 3,

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+$$
$$\partial_t E + \mathrm{div}(u(E + p)) = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+$$

$E = \frac{1}{2}\rho|u|^2 + \rho e$
$p = \varphi(\rho, e)$ (perfect gaz : $p = (\gamma - 1)\rho e$)
Initial condition on ρ, u, p

Academic Models: Burgers Equation, Transport Equation

Burgers: $\partial_t \rho + \partial_x(\rho u) = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+,\ u = \rho$

Transport: $\partial_t \rho + \partial_x(c\rho) = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+,\ c \in \mathbb{R}, \textit{given}$

Initial condition on ρ

Numerical Analysis: behaviour of numerical schemes, stability, convergence. . .

Numerical Analysis for some schemes

Numerical analysis (stability, convergence…) may (perhaps) help to choose the numerical scheme

- Upwinding
- Using staggered grids
- Working with an “equivalent” equation

Explicit upwind scheme for transport equation

$$\partial_t \rho + \partial_x \rho = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$

time step : k, $t_n = nk$, $n \in \mathbb{N}$, mesh size : h
Mesh : M_i, $i \in \mathbb{Z}$
$M_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$, $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = h$
center of M_i : x_i
ρ_i^n is supposed to be an approximate value for $\rho(x_i, t_n)$

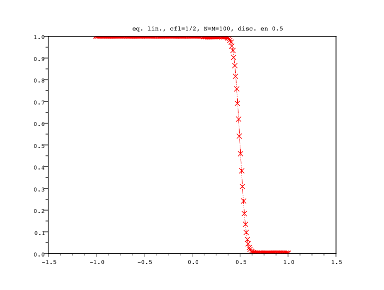
$$\frac{1}{k}(\rho_i^{n+1} - \rho_i^n) + \frac{1}{h}(\rho_i^n - \rho_{i-1}^n) = 0$$

$\frac{k}{h}$ is the CFL number

Upwind scheme for transport equation

$$\partial_t \rho + \partial_x \rho = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$
$$\rho(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

Upwind scheme, CFL=1/2, solution for T=1/2 ($N = M = 100$)
space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h$

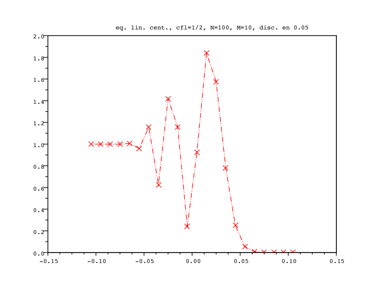


Good speed of discontinuity, bounds on the solution, large amount of numerical diffusion

Why upwinding ?

Centered scheme, CFL=1/2, solution for T=1/20 ($N = 100$, $M = 10$)

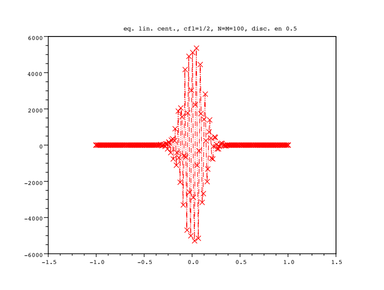
$$\frac{1}{k}(\rho_i^{n+1} - \rho_i^n) + \frac{1}{2h}(\rho_{i+1}^n - \rho_{i-1}^n) = 0$$



no numerical diffusion but oscillations, no convergence.

Why upwinding ?

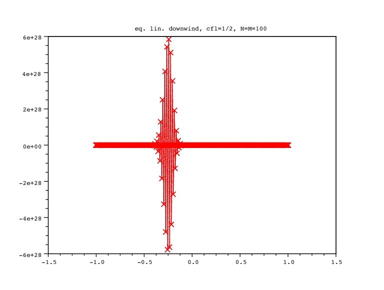
Centered scheme, CFL=1/2, solution for T=1/2 ($N = 100$, $M = 100$).



no numerical diffusion but oscillations, no convergence.

Downwind scheme, for joke

Downwind scheme, CFL=1/2, solution for T=1/2 ($N = 100$, $M = 100$).



numerical antidiffusion, no convergence.

Burgers, upwind

$$\partial_t \rho + \partial_x (u\rho) = 0, \quad u = \rho, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$
$$\rho(x,0) = \begin{cases} 2,\ x < 0 \\ 1,\ x > 0 \end{cases}$$

Upwind scheme
Space step: h, time step : $k = (CFL)h/4$

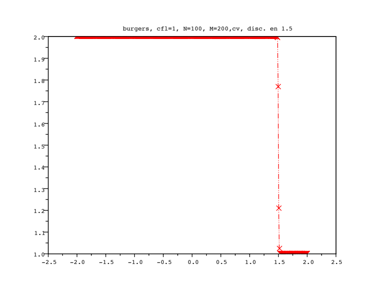
$$\frac{1}{k}(\rho_i^{n+1} - \rho_i^n) + \frac{1}{h}((\rho_i^n)^2 - (\rho_{i-1}^n)^2) = 0$$

$f(\rho) = \rho^2$, $4 = \max\{f'(s),\ 1 \le s \le 2\}$

Burgers, upwind

$$\partial_t \rho + \partial_x(u\rho) = 0, \quad u = \rho, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$
$$\rho(x,0) = \begin{cases} 2, & x < 0 \\ 1, & x > 0 \end{cases}$$

Upwind scheme, CFL=1, solution for T=1/2 ($N = 100$, $M = 200$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/4$



Good localization of the discontinuity, few numerical diffusion, bounds on the solution, convergence to the entropy solution

Burgers , upwind-ncv

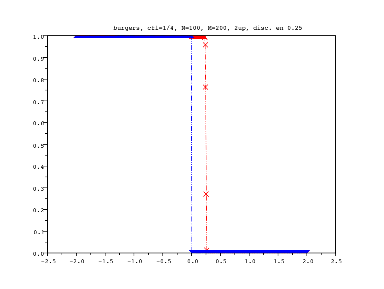
Upwind on $u\partial_x\rho + \rho\partial_x u$. Since $u = \rho$ (collocated), it gives

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + 2u_i^n(\rho_i^n - \rho_{i-1}^n) = 0, \quad u_i^n = \rho_i^n$$

Initial condition : 1 for $x < 0$ and 0 for $x > 0$

Upwind-ncv scheme, CFL=1/4, solution for T=1/4 ($N = 100$, $M = 200$)

Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/2$



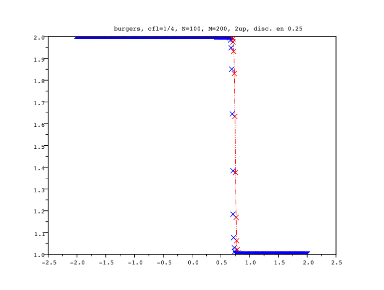
Wrong localization of the discontinuity (0 instead of 0.25 !), no numerical diffusion !, bounds on the solution, no convergence.

Burgers, upwind-ncv

Upwind on $u\partial_x\rho + \rho\partial_x u$. Since $u = \rho$ (collocated), it gives

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + 2u_i^n(\rho_i^n - \rho_{i-1}^n) = 0, \quad u_i^n = \rho_i^n$$

Initial condition : 2 for $x < 0$ and 1 for $x > 0$
Upwind-ncv scheme, CFL=1, solution for T=1/4 ($N = 100$, $M = 200$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/4$



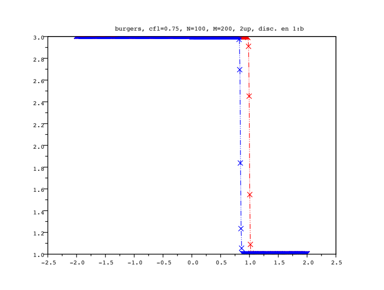
not so bad, curious result...due to this particular initial condition

Burgers viewed as a coupled system, upwind-ncv

Upwind on $u\partial_x \rho + \rho \partial_x u$. Since $u = \rho$ (collocated), it gives

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + 2u_i^n(\rho_i^n - \rho_{i-1}^n) = 0, \quad u_i^n = \rho_i^n$$

Initial condition : 3 for $x < 0$ and 1 for $x > 0$
Upwind-ncv scheme, CFL=1/4, solution for T=1/4 ($N = 100$, $M = 200$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/2$



Wrong localization of the discontinuity, bounds on the solution, no convergence.

Burgers viewed as a coupled system, upwind-ncv

Upwind on $u\partial_x \rho + \rho\partial_x u$ (or $2\rho\partial_x \rho$)

Upwind-ncv=upwind + discretization of $h(\partial_x u)^2$.

No problem for a regular solution. A problem might arise if $\partial_x u$ not in L^2.

- Full upwind collocated scheme is perfect. Good discontinuity, bounds on the solution, convergence
- Non conservative upwind collocated scheme is not good.

What happens with staggered grids?

Staggered grid

time step : k, $t_n = nk$, $n \in \mathbb{N}$, mesh size : h
Mesh : M_i, $i \in \mathbb{Z}$
$M_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$, $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = h$
center of M_i : x_i
ρ_i^n is supposed to be an approximate value for $\rho(x_i, t_n)$
$u_{i+\frac{1}{2}}^n$ is supposed to be an approximate value for $u(x_{i+\frac{1}{2}}, t_n)$

Discretization of $u\rho$ at point (x_i, t_n) is $\frac{1}{h}(u_{i+\frac{1}{2}}^n \rho_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n \rho_{i-\frac{1}{2}}^n)$
Choice of $\rho_{i+\frac{1}{2}}^n$ and $\rho_{i-\frac{1}{2}}^n$?

Burgers, upwind-staggered

$$\partial_t \rho + \partial_x(u\rho) = 0, \quad u = \rho, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$
$$\rho(x,0) = \begin{cases} 2, & x < 0 \\ 1, & x > 0 \end{cases}$$

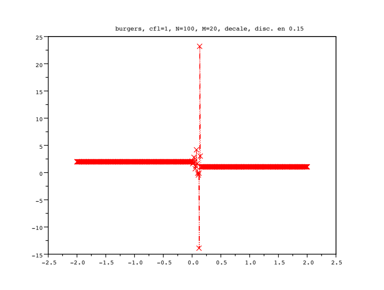
Upwind staggered scheme:
We have $\rho > 0$ (and then $u > 0$).
Then upwind staggered consists to take $\rho^n_{i+\frac{1}{2}} = \rho^n_i$

Space step: h, time step: k
Since the exact solution is between 1 and 2, it seems that we can take $k = (CFL)h/4$ with $CFL \leq 1$

Burgers, upwind-staggered

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + (u^n_{i+\frac{1}{2}}\rho_i^n - u^n_{i-\frac{1}{2}}\rho^n_{i-1}) = 0,$$
$$u_{i+\frac{1}{2}} = (1/2)(\rho_i^n + \rho^n_{i+1})$$

Upwind-staggered scheme, CFL=1, solution for T=1/20 ($N = 100$, $M = 20$)
Space step: $h = 1/N$, M = number of time steps, $k = (CFL)h/4$

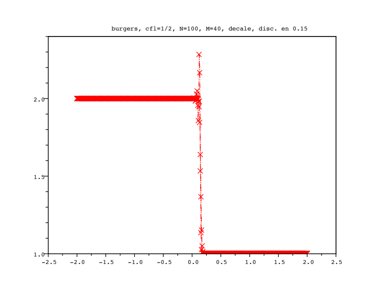


Pretty good localization of the discontinuity (0.15), but no bound of the solution $\rightsquigarrow$ time step too large

Burgers viewed as a coupled system, upwind-staggered

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + (u^n_{i+\frac{1}{2}}\rho_i^n - u^n_{i-\frac{1}{2}}\rho^n_{i-1}) = 0,$$
$$u_{i+\frac{1}{2}} = (1/2)(\rho_i^n + \rho_{i+1}^n)$$

Upwind-staggered scheme, CFL=1/2(reduced CFL), solution for T=1/20 ($N = 100$, $M = 40$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/4$

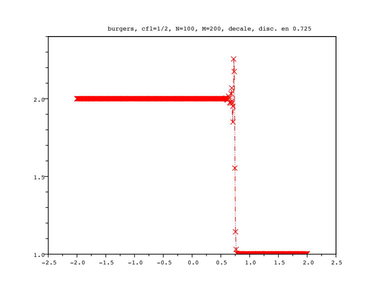


Good localization of the discontinuity (0.15), positivity but no upper bound on the solution.

Burgers viewed as a coupled system, upwind-staggered

$$(h/k)(\rho_i^{n+1} - \rho_i^n) + (u^n_{i+\frac{1}{2}}\rho_i^n - u^n_{i-\frac{1}{2}}\rho^n_{i-1}) = 0,$$
$$u_{i+\frac{1}{2}} = (1/2)(\rho_i^n + \rho^n_{i+1})$$

Upwind-staggered scheme, CFL=1/2(reduced CFL), solution for T=1/4 ($N = 100$, $M = 200$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/4$



Good localization of the discontinuity (0.75), positivity but no upper bound on the solution.

Burgers viewed as a coupled system

$$\partial_t \rho + \partial_x(u\rho) = 0, \quad u = \rho, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$

$$\rho(x,0) = \begin{cases} 2, & x < 0 \\ 1, & x > 0 \end{cases}$$

Upwind scheme with staggered grids is pretty good... Good discontinuity, positivity of the solution, no upper bound (and then reduced CFL is needed) but probably convergence

Work in progress: D. Doyen and R. Eymard

Burgers view as a coupled system, conclusion

$$\partial_t \rho + \partial_x (u\rho) = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$

$$u = \rho, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+$$

with a discontinuous solution

possible schemes

- Collocated scheme, full upwind (upwind on $u\rho$)
- staggered, upwind on ρ
- (staggered, upwind on u and ρ)

It is possible to use staggered schemes for compressible fluid mechanics

Is it possible to work with an "equivalent equation"?

Why working on an equivalent equation?

Example: Euler system is equivalent, for regular solutions, to the following one

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \\ \partial_t \rho e + \mathrm{div}(\rho u e) + p\,\mathrm{div}\, u &= 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \end{aligned}$$

and there are some reasons to prefer (in particular with staggered grids) to work with this system instead of the initial system

But, this system is not equivalent to the initial system when the solution contains shocks

Burgers, upwind on an "equivalent equation"

$$\partial_t \rho + \partial_x(\rho^2) = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}_+$$
$$\rho(x,0) = \begin{cases} 2, \; x < 0 \\ 1, \; x > 0 \end{cases}$$

For positive and regular solution, an equivalent equation is

$$\partial_t \rho^2 + \frac{4}{3}\partial_x(\rho^3) = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}_+$$

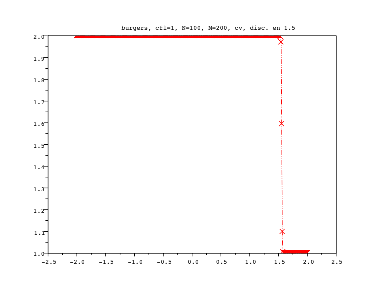
The classical upwind scheme on this latter equation leads to a solution which does not have the good localization of the discontinuity

The speed of the discontinuity is 3 for burgers and $28/9$ for the equivalent equation

Burgers, upwind on an "equivalent" equation

$$(h/k)((\rho_i^{n+1})^2 - (\rho_i^n)^2) + \frac{4}{3}((\rho_i^n)^3 - (\rho_{i-1}^n)^3) = 0,$$

Upwind scheme on the "equivalent" equation, CFL=1, solution for T=1/2 ($N = 100$, $M = 200$)
Space step: $h = 1/N$, $M =$ number of time steps, $k = (CFL)h/4$



Bad localization of the discontinuity (1.555 instead of 1.5), bounds on the solution, no convergence

Burgers, numerical diffusion

$$\partial_t \rho + \partial_x(f(\rho)) = 0$$

On this equation, if $f' \geq 0$, upwinding is "similar" to add a numerical diffusion. Namely, is similar to

$$\partial_t \rho + \partial_x(f(\rho)) - \partial_x(\frac{hf'(\rho) - kf'^2(\rho)}{2}\partial_x \rho) = 0$$

The CFL condition is for $hf'(\rho) - kf'^2(\rho) \geq 0$ (i.e. $kf'(\rho) \leq h$)

In the case of the Burgers equation it gives

$$\partial_t \rho + \partial_x(\rho^2) - \partial_x((h\rho - 2k\rho^2)\partial_x \rho) = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}_+$$

Burgers, non conservative numerical diffusion

In the case of the “equivalent” equation

$$\partial_t \rho^2 + (4/3)\partial_x(\rho^3) = 0,$$

upwinding is similar to (since $\rho > 0$)

$$\partial_t \rho^2 + \frac{4}{3}\partial_x(\rho^3) - \partial_x((2h\rho^2 - 4k\rho^3)\partial_x \rho) = 0,$$

Turning back to the Burgers equation, this leads to

$$\partial_t \rho + \partial_x(\rho^2) - \frac{1}{\rho}\partial_x((h\rho^2 - 2k\rho^3)\partial_x \rho) = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}_+$$

This is a numerical diffusion (thanks to the CFL condition) but not on a conservative form.

The consequence is that a non conservative diffusion may lead to wrong discontinuities

Burgers, non conservative numerical diffusion on an equivalent equation

The discretization of a non conservative diffusion on the burgers equation lead to wrong discontinuities

But

Using a non conservative diffusion on an equivalent equation may gives the good discontinuities for the initial equation?

The answer is yes. . . (T. Gallouët, R. Herbin, J.-C. Latché and T. T. Trung)

Working with internal energy in Euler Equations

when the solution contains a shock wave, the initial Euler Equations are not equivalent to the following ones

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \\ \partial_t \rho e + \operatorname{div}(\rho u e) + p \operatorname{div} u = 0,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \end{aligned}$$

But, discretizing the third equation by adding a convenient source term gives an approximate solution which converges, as the mesh size and the time step go to 0 (with a *CFL* condition in the case of an explicit scheme), to a weak solution of the Euler Equations (assuming some estimates on the approximate solution).

Papers of R. Herbin, W. Kheriji, J.-C. Latché and T. T. Trung

Indeed, the source term converge to 0 except in the shocks waves.

Stationary compressible Stokes equations

Works with R. Eymard, A. Fettah, R. Herbin and J. C. Latché.

$d = 2$ or 3, Ω bounded domain of $\mathbb{R}^d$, $d = 2$ or $d = 3$
$f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) dx = M,$$

$$p = \varphi(\rho) \text{ in } \Omega, \, \varphi \text{ increasing and } \liminf_{s \to +\infty} \frac{\varphi(s)}{s} = +\infty.$$

(Example: $p = \rho^\gamma$, $\gamma > 1$)

Discretization by schemes with staggered grids

Main result

- Two possible discretizations for the momentum equation :
  $\rightsquigarrow$ MAC scheme (most commonly used scheme for incompressible Navier Stokes equations)
  $\rightsquigarrow$ Crouzeix-Raviart Finite Element
- Discretization of the mass equation (and total mass constraint) by classical upwind Finite Volume
- Existence of solution for the discrete problem
- Proof of the convergence (up to subsequence) of the solution of the discrete problem towards a weak solution of the continuous problem (no uniqueness result for this problem) as the mesh size goes to 0

Generalizations

- (Easy) Complete Stokes problem:
  $-\mu\Delta u - \frac{\mu}{3}\nabla(\operatorname{div} u) + \nabla p = f$, with $\mu \in \mathbb{R}_+^\star$ given
- (Ongoing work) Navier-Stokes Equations with $\gamma > 1$ if $d = 2$ and $\gamma > \frac{3}{2}$ if $d = 3$ (probably sharp result with respect to γ without changing the diffusion term or the EOS)
- (Open question) Other boundary condition. Addition of an energy equation
- (Ongoing work) Evolution equation (Stokes and Navier-Stokes), convergence and error estimates (in MODTERCOM...)