Convergence and error estimates for bounded numerical solutions of the barotropic Navier-Stokes system

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Abstract

We consider a mixed finite-volume finite-element method applied to the Navier-Stokes system of equations describing the motion of a compressible, barotropic, viscous fluid. We show convergence as well as error estimates for the family numerical solutions on condition that:

- the underlying physical domain as well as the data are smooth;
- the time step Δt and the parameter of the parameter h of the spatial discretization are proportional, $\Delta t \approx h$;
- the family of numerical densities remains bounded for $\Delta t, h \to 0$.

No *a priori* smoothness is required for the limit (exact) solution.

Key words: Navier-Stokes system, mixed numerical method, convergence, error estimates

1 Introduction

We study a numerical approximation of the Navier-Stokes system in a space-time cylinder $Q_T = (0,T) \times \Omega$, where $T > 0$ is arbitrary and $\Omega \subset R^3$ is a bounded domain, where the fluid density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ satisfy

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

with the viscous stress tensor $\mathbb{S}$ given by Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \ \mu > 0. \tag{1.3}$$

Equations (1.1–1.2) are supplemented with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{1.4}$$

[*]The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[†]The research of R.H. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[‡]The work of D.M. and A.N. has been supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region

and initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \text{ in } \Omega. \tag{1.5}$$

For the sake of simplicity, we have deliberately omitted the effect of external forces in (1.2). We also adopted the so-called Stokes' hypothesis taking the bulk viscosity to be zero in (1.3). As the shear viscosity coefficient μ is constant, we may write

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mu \Delta \mathbf{u} + \frac{\mu}{3} \nabla_x \mathrm{div}_x \mathbf{u}. \tag{1.6}$$

Finally, we suppose very mild and physically grounded hypotheses concerning the pressure, namely

$$p \in C^3(0,\infty) \cap C^1[0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho \geq 0. \tag{1.7}$$

Moreover, the assumption $p'(0) > 0$ can be relaxed at the expense of some additional technicalities in the proofs.

We consider a family of approximate solutions $\{\varrho_h, \mathbf{u}_h\}_{h>0}$ constructed via the numerical scheme proposed by Karper [12] (see also Karlsen and Karper [9], [10], [11]), with the necessary modifications introduced in [4] to accommodate smooth fluid domains. Our goal is to show convergence and qualitative error estimates for the numerical solutions on condition that:

- the physical domain Ω as well as the initial data $[\varrho_0, \mathbf{u}_0]$ are sufficiently smooth;
- the time step Δt and the parameter of the parameter h of the spatial discretization are proportional, $\Delta t \approx h$;
- the family $\{\varrho_h\}_{h>0}$ of approximate densities remains bounded for $h \to 0$.

We point out that, in contrast with the standard *a priori* error estimates commonly studied in the numerical literature, see e.g. Liu [14], [15], our result does not require any information about the smoothness of the exact solution that will in fact follow as a byproduct of the proof. Our approach leans on the following results:

- *convergence* of the underlying numerical scheme established by Karper [12], with the extension to smooth domains studied in [4];
- *regularity criterion* for (exact) solutions of the compressible Navier-Stokes system shown by Sun, Wang, and Zhang [16];
- a discrete version of the *relative energy inequality* for the Navier-Stokes system obtained by Gallouët at al [7];
- *a priori* error estimates for the Navier-Stokes system derived in [2].

Before passing to rigorous and mostly very technical mathematical statements, we present some heuristic arguments underlying our strategy:

- We consider the numerical solutions $\{\varrho_h, \mathbf{u}_h\}_{h>0}$ constructed by Karper et al. [12], [4] on an *unfitted* mesh $\Omega_h \approx \Omega$. By virtue of the convergence result established in [4], we obtain a subsequence such that

  $$\varrho_h \to \varrho,\ \mathbf{u}_h \to \mathbf{u} \text{ in a certain sense specified below}, \tag{1.8}$$

  where $[\varrho, \mathbf{u}]$ is a *weak solution* of problem (1.1–1.5).
- As the initial data as well as the physical domain Ω of the limit problem are smooth, there exists a strong solution $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ emanating from the initial data $[\varrho_0, \mathbf{u}_0]$ defined on a (maximal) time interval $[0, \tilde{T})$, $\tilde{T} \leq T$. In view of the weak-strong uniqueness property shown in [3], the weak solution $[\varrho, \mathbf{u}]$ *coincides* with $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ on $[0, \tilde{T})$.

- In accordance with our principal hypothesis, the density components ϱ_h of numerical solutions are uniformly bounded for $h \to 0$; whence

$$\varrho \in L^\infty((0,T) \times \Omega).$$

Now, we can use the conditional regularity result of Sun, Wang, and Zhang [16] claiming that the strong solution $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ of problem (1.1–1.5) remains smooth as long as its density component is bounded, which yields, in particular, $\tilde{T} = T$ and $\varrho = \tilde{\varrho}$, $\mathbf{u} = \tilde{\mathbf{u}}$ in $(0,T) \times \Omega$. In particular, the limit solution is uniquely determined by the initial data and there is no need of subsequence in (1.8).

- Knowing that the limit solution is smooth, we can mimick step by step the estimates elaborated in Gallouët et al. [6] to establish the desired error estimates. Note that some steps performed in [6] must be modified in the spirit of [2] in order to control the approximation error for the unfitted mesh used in the present paper.

The paper is organized as follows. In Section 2, we introduce the numerical method to construct the approximate solutions. Our main result is formulated in Section 3. In Section 4, we study convergence of the approximate solutions and establish regularity properties of the limit. In Section 5 we introduce the main tool used in the proof - the relative energy inequality. The proof of the error estimates is completed in Section 6.

2 Numerical method

Problem (1.1–1.5) will be solved by means of the numerical method proposed in [5] based on time discretization, finite-volume discretization of the convective terms (upwind), and a finite-element discretization of the viscous stress in (1.2). We shall write

$$a \stackrel{<}{\sim} b \text{ if } a \leq cb,\ c > 0 \text{ a constant},\ a \approx b \text{ if } a \stackrel{<}{\sim} b \text{ and } b \stackrel{<}{\sim} a.$$

Here, "constant" typically means a generic quantity independent of the size of the mesh and the time step used in the numerical scheme as well as other parameters as the case may be.

2.1 Mesh, domain approximation

We consider a family of numerical domains Ω_h,

$$\Omega_h = \cup_{E \in E_h} E,$$

where E_h denotes a *tetrahedral* mesh with individual (compact) elements E. Faces in the mesh are denoted as Γ, whereas Γ_h is the set of all faces. Moreover, the set of faces $\Gamma \subset \partial\Omega_h$ is denoted $\Gamma_{h,\text{ext}}$, while $\Gamma_{h,\text{int}} = \Gamma_h \setminus \Gamma_{h,\text{ext}}$.

We require the mesh to be *shape regular*, specifically:

- The intersection $E \cap F$ of two elements $E, F \in E_h$ is either empty, or their common face, or their common edge, or their common vortex.

- The diameter diam$[E]$ of each element is proportional to $h > 0$,

$$\text{diam}[E] \approx h \text{ for any } E \in E_h.$$

- The radius of the largest ball $r[E]$ contained in E is also proportional to h,

$$r[E] \approx h \text{ for any } E \in E_h.$$

Finally, we suppose that the family Ω_h approaches the physical domain Ω in the sense that

$$\inf_{x\in\partial\Omega} |y-x| \lesssim h \text{ uniformly for all } y \in \partial\Omega_h. \tag{2.1}$$

Remark 2.1. *It is easy to see that the approximation property (2.1) is satisfied for the so-called unfitted mesh E_h, obtained as*

$$E \in E_h \text{ if } E \cap \overline{\Omega} \neq \emptyset,$$

where the elements E belong to a shape regular mesh $\tilde{E}_h$ filling a large domain B containing Ω in its interior. One can even take $B = R^3$ - examples of shape regular meshes of elements enjoying further specific geometric properties were constructed e.g. by Hošek [8], Vanderzee et al. [17].

2.2 Finite volumes, finite elements, upwind

Each face $\Gamma \in \Gamma_h$ is associated with a (fixed) normal vector $\mathbf{n}$. We write Γ_E whenever a face $\Gamma_E \subset \partial E$ is considered as a part of the boundary of the element E. In such a case, the normal vector to Γ_E is always the *outer* normal vector with respect to E. Keeping this convention in mind we introduce for any function g, continuous on each element E,

$$g^{\text{out}}|_\Gamma = \lim_{\delta\to 0+} g(\cdot + \delta\mathbf{n}),\ g^{\text{in}}|_\Gamma = \lim_{\delta\to 0+} g(\cdot - \delta\mathbf{n}),\ [[g]]_\Gamma = g^{\text{out}} - g^{\text{in}}, \{g\}_\Gamma = \frac{1}{2}\left(g^{\text{out}} + g^{\text{in}}\right). \tag{2.2}$$

For $\Gamma_E \subset \partial E$ we simply write g for g^{in}. Occasionally, we also omit the subscript Γ if no confusion arises.

2.2.1 Spaces of piecewise constant functions

We introduce the space of piecewise constant functions

$$Q_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \;\middle|\; v|_E = a_E \in R \text{ for any } E \in E_h \right\},$$

with the associated projection

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h),\ \Pi_h^Q[v]|_E = \frac{1}{|E|}\int_E v \ \mathrm{d}x. \tag{2.3}$$

To keep the notation concise, we will occasionally denote

$$\Pi_h^Q[v] \equiv \hat{v}.$$

2.2.2 Crouzeix-Raviart finite elements

A discrete counterpart D_h of a differential operator D acting in the $x-$variable is

$$D_h v|_E = D(v|_E) \text{ for any } v \text{ differentiable on each element } E \in E_h.$$

The *Crouzeix-Raviart finite element spaces* (see e.g. Brezzi and Fortin [1]) are defined as

$$V_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \;\middle|\; v|_E = \text{affine function},\ E \in E_h,\ \int_\Gamma [[v]] \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\text{int}} \right\}. \tag{2.4}$$

In view of the no-slip boundary condition (1.4), we will also need the space

$$V_{h,0}(\Omega_h) = \left\{ v \in V_h \;\middle|\; \int_\Gamma v \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\text{ext}} \right\}. \tag{2.5}$$

The associated projection is

$$\Pi_h^V : W^{1,q}(\Omega_h) \to V_h(\Omega_h) \text{ requiring } \int_\Gamma \Pi_h^V[v] \ \mathrm{dS}_x = \int_\Gamma v \ \mathrm{dS}_x \text{ for any } \Gamma \in \Gamma_h. \tag{2.6}$$

2.2.3 Upwind

Denote

$$[c]^+ = \max\{c,0\},\ [c]^- = \min\{c,0\},\ \langle v\rangle_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma v\ \mathrm{dS}_x.$$

Following [4], we introduce a *dissipative upwind* operator $\mathrm{Up}[r,\mathbf{u}]$ on a face Γ by

$$\mathrm{Up}[r,\mathbf{u}] = \frac{r^{\mathrm{in}}}{2}\left([\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma + h^\alpha]^+ + [\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma - h^\alpha]^+\right) + \frac{r^{\mathrm{out}}}{2}\left([\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma + h^\alpha]^- + [\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma - h^\alpha]^-\right), \tag{2.7}$$

with a positive exponent α determined below. Note that such a definition makes sense as soon as $r\in Q_h(\Omega_h)$, $\mathbf{u}\in V_h(\Omega_h;R^3)$ and $\Gamma\in\Gamma_{h,\mathrm{int}}$.

Setting, formally, $h^\alpha\approx 0$ in (2.7), we obtain the conventional definition of the upwind operator

$$r^{\mathrm{in}}[\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma]^+ + r^{\mathrm{out}}[\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma]^-.$$

To see the dissipative character of this upwind operator, we write

$$\mathrm{Up}[r,\mathbf{u}] = \underbrace{r^{\mathrm{in}}[\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma]^+ + r^{\mathrm{out}}[\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma]^-}_{\text{conventional upwind}} - \underbrace{[[r]]_\Gamma\, h^\alpha\chi\left(\frac{\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma}{h^\alpha}\right)}_{\text{dissipative component}}, \tag{2.8}$$

where

$$\chi(z) = \begin{cases} 0 \text{ for } z<-1,\\ \frac{1}{2}(z+1) \text{ if } -1\le z\le 0,\\ -\frac{1}{2}(z-1) \text{ if } 0<z\le 1,\\ 0 \text{ for } z>1,\end{cases}$$

and where the dissipative component is reminiscent of the finite volume discretization of the conventional artificial diffusion operator $-h^\alpha\Delta r$.

2.3 Numerical scheme

Extending both ϱ_0 and $\mathbf{u}_0$ to be zero outside Ω we set

$$\varrho_h^0 = \Pi_h^Q[\varrho_0]\in Q_h(\Omega_h),\ \mathbf{u}_h^0 = \Pi_h^Q[\mathbf{u}_0]\in Q_h(\Omega_h;R^3). \tag{2.9}$$

Next, we introduce the discrete time derivative

$$D_t b_h^k = \frac{b_h^k - b_h^{k-1}}{\Delta t},\ \Delta t\approx h,$$

and define (implicitly) a family of numerical solutions $\{\varrho_h^k,\mathbf{u}_h^k\}_{h>0}$, $k=1,2,\dots,$

$$\varrho_h^k\in Q_h(\Omega_h),\ \mathbf{u}_h^k\in V_{h,0}(\Omega_h;R^3)$$

satisfying:

$$\int_{\Omega_h} D_t\varrho_h^k\phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k,\mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x = 0 \text{ for all } \phi\in Q_h(\Omega_h), \tag{2.10}$$

$$\int_{\Omega_h} D_t(\varrho_h^k\hat{\mathbf{u}}_h^k)\cdot\phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k\hat{\mathbf{u}}_h^k,\mathbf{u}_h^k]\cdot\left[\left[\widehat{\phi}\right]\right]\ \mathrm{dS}_x \tag{2.11}$$

$$+\int_{\Omega_h}\left[\mu\nabla_h\mathbf{u}_h^k:\nabla_h\phi + \frac{\mu}{3}\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_h\phi\right]\ \mathrm{d}x - \int_{\Omega_h} p(\varrho_h^k)\mathrm{div}_h\phi\ \mathrm{d}x = 0 \text{ for all } \phi\in V_{h,0}(\Omega_h;R^3).$$

3 Main result

To formulate our main result, it is convenient to extend the numerical solution to be defined for *any* $t \in (-\infty, T)$. To this end, we set

$$\varrho_h(t,\cdot) = \varrho_h^0,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t \leq 0,$$

$$\varrho_h(t,\cdot) = \varrho_h^k,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1,2,\dots. \tag{3.1}$$

Accordingly, we define

$$D_t v_h(t,\cdot) = \frac{v_h(t) - v_h(t-\Delta t)}{\Delta t},\ t > 0.$$

Our main result reads as follows:

Theorem 3.1. *Let $\Omega \subset R^3$ be a bounded domain of class C^3. Let the initial data $[\varrho_0, \mathbf{u}_0]$ belong to the regularity class*

$$\varrho_0 \in C^3(\overline{\Omega}),\ \varrho_0 > 0 \text{ in } \overline{\Omega},\ \mathbf{u}_0 \in C^3(\overline{\Omega}; R^3),$$

and satisfy the compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x p(\varrho_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_0)|_{\partial\Omega}.$$

Let $\{\varrho_h^k, \mathbf{u}_h^k\}_{h>0}$, $k = 0,1,\dots,[T/\Delta t]$, $h \approx \Delta t$, be a family of numerical solutions satisfying (2.10), (2.11), where the upwind term is determined by (2.7), with

$$0 < \alpha < 1.$$

Finally, suppose that

$$\varrho_h^k \leq r < \infty \text{ for all } h > 0,\ k = 0,1,\dots,[T/\Delta t]. \tag{3.2}$$

Then problem (1.1–1.5) admits a classical solution $[\varrho, \mathbf{u}]$ in $(0,T)\times\Omega$, and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega\cap\Omega_h} \left[\varrho_h |\widehat{\mathbf{u}}_h - \mathbf{u}|^2 + |\varrho_h - \varrho|^2\right](t,\cdot)\ \mathrm{d}x + \int_0^T \int_{\Omega\cap\Omega_h} |\nabla_h \mathbf{u}_h - \nabla_x \mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t \tag{3.3}$$

$$\lesssim \left(h^{\alpha/2} + \int_{\Omega_h} \left[\varrho^0 |\widehat{\mathbf{u}}^0 - \mathbf{u}_0|^2 + |\varrho^0 - \varrho_0|^2\right]\ \mathrm{d}x \right).$$

Remark 3.1. *We point out that the* existence *of the classical exact solution $[\varrho, \mathbf{u}]$ is not assumed a priori. As we will see in the next section, it is a consequence of hypothesis (3.2) and the convergence result for the numerical scheme established in [4].*

Remark 3.2. *Since the Crouzeix-Raviart elements are of the first order, the rate of convergence stated in (3.3) is optimal also in view of the approximation distance between Ω_h and Ω stated in (2.1), cf. Lenoir [13].*

The rest of the paper is devoted to the proof of Theorem 3.1.

4 Convergence, regularity of the limit solution

Since the numerical densities are assumed to be uniformly bounded, the behavior of the pressure $p = p(\varrho)$ for large values of ϱ is irrelevant. In particular, we may take pressure $p(\varrho) \approx \varrho^\gamma$ for large values of ϱ, with $\gamma > 1$ arbitrary. Consequently, we may apply [4, Theorem 3.1] to obtain the following conclusion:

Under the hypotheses of Theorem 3.1, extending ϱ_h, $\mathbf{u}_h$ to be zero outside Ω_h, we may extract a subsequence of the numerical solutions such that

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho, \mathbf{u}]$ is a weak solution of problem (1.1 - 1.5) in the space time cylinder $(0,T)\times\Omega$. Moreover, as a consequence of hypothesis (3.2),

$$\varrho \in L^\infty((0,T)\times\Omega). \tag{4.1}$$

Now, since the limit density is bounded, the data as well as the spatial domain are regular, we may use the conditional regularity result of Sun, Wang, and Zhang [16], together with the weak-strong uniqueness principle established in [3], to conclude that the limit solution is regular; whence unique. More specifically, $[\varrho, \mathbf{u}]$ is a classical solution of problem (1.1–1.5), the density ϱ is positive bounded below away from zero, and

$$\|1/\varrho\|_{C([0,T]\times\overline{\Omega})} + \|\varrho\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla_x \varrho\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t}\varrho\|_{C([0,T];L^6(\Omega))} \leq D, \tag{4.2}$$

$$\|\mathbf{u}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{u}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{u}\|_{C([0,T];L^6(\Omega;R^{3\times3}))} + \|\partial^2_{t,t}\mathbf{u}\|_{C([0,T];L^6(\Omega;R^3))} \leq D, \tag{4.3}$$

where D is a constant depending only on T and the regularity properties of the initial data, see [2, Proposition 2.1].

5 Relative energy

Having observed that the exact solution is smooth, we are ready to derive rigorous error estimates for the approximate solutions. Following [6] we evaluate the differences $\varrho_h - \varrho$, $\mathbf{u}_h - \mathbf{u}$ by means of the *relative energy functional*

$$\mathcal{E}\left(\varrho_h, \mathbf{u}_h \middle| \varrho, \mathbf{u}\right) = \int_{\Omega_h} \left[\frac{1}{2}\varrho_h|\mathbf{u}-\mathbf{u}_h|^2 + H(\varrho_h) - H'(\varrho)(\varrho_h - \varrho) - H(\varrho)\right]\ \mathrm{d}x,$$

where H is the pressure potential,

$$H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

In accordance with hypothesis (1.7), the pressure $p = p(\varrho)$ is strictly increasing, therefore $H = H(\varrho)$ is a strictly convex function. Moreover, as ϱ_h, ϱ range in a bounded subset of $[0,\infty)$ and ϱ is strictly positive, we may assume

$$H(\varrho_h) - H'(\varrho)(\varrho_h - \varrho) - H(\varrho) \approx (\varrho_h - \varrho)^2. \tag{5.1}$$

5.1 Discrete relative energy inequality

The abstract form of the inequality satisfied by the relative energy functions $\mathcal{E}$ was derived in [3]. Here, we introduce its discrete analogue proved in Gallouet et al. [7]:

$$\mathcal{E}\left(\varrho_h^n, \hat{\mathbf{u}}_h^n \middle| r_h^n, \hat{\mathbf{v}}_h^n\right) + \Delta t\sum_{k=1}^n \left(\mu\int_{\Omega_h} |\nabla_h \mathbf{u}_h^k - \nabla_h \mathbf{v}_h^k|^2\ \mathrm{d}x + \frac{\mu}{3}\int_{\Omega_h} |\mathrm{div}_h \mathbf{u}_h^k - \mathrm{div}_h \mathbf{v}_h^k|^2\ \mathrm{d}x\right) \tag{5.2}$$

$$\lesssim \mathcal{E}\left(\varrho_h^0, \widehat{\mathbf{u}}_h^0 \middle| r_h^0 \widehat{\mathbf{v}}_h^0\right) + \Delta t \sum_{k=1}^{n} \sum_{j=1}^{6} \mathcal{R}_{h,j}^k,$$

for any $r_h^k \in Q_h(\Omega_h)$, $r_h^k > 0$, $\mathbf{v}_h^k \in V_{h,0}(\Omega_h; R^3)$, where the remainders are

$$\mathcal{R}_{h,1}^k = \mu \int_{\Omega_h} \nabla_h \mathbf{v}_h^k : \nabla_h \left(\mathbf{v}_h^k - \mathbf{u}_h^k\right) \ \mathrm{d}x + \frac{\mu}{3} \int_{\Omega_h} \mathrm{div}_h \mathbf{v}_h^k \mathrm{div}_h \left(\mathbf{v}_h^k - \mathbf{u}_h^k\right) \ \mathrm{d}x,$$

$$\mathcal{R}_{h,2}^k = \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t}\right) \cdot \left(\frac{\mathbf{v}_h^k + \mathbf{v}_h^{k-1}}{2} - \mathbf{u}_h^{k-1}\right) \ \mathrm{d}x$$

$$\mathcal{R}_{h,3}^k = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \left[\left[\widehat{\mathbf{v}}_h^k\right]\right] \ \mathrm{dS}_x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \left\{\widehat{\mathbf{v}}_h^k\right\} \left[\left[\widehat{\mathbf{v}}_h^k\right]\right] \ \mathrm{dS}_x$$

$$\mathcal{R}_{h,4}^k = -\int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \mathbf{v}_h^k \ \mathrm{d}x,$$

$$\mathcal{R}_{h,5}^k = \int_{\Omega_h} \frac{H'(r_h^k) - H'(r_h^{k-1})}{\Delta t} (r_h^k - \varrho_h^k) \ \mathrm{d}x,$$

$$\mathcal{R}_{h,6}^k = \sum_{G \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \left[\left[H'(r_h^{k-1})\right]\right] \ \mathrm{dS}_x.$$

5.2 Extending the exact solution

The leading idea of the proof of the error estimates (3.3) is now the same as in [2], namely to take r_h^k, $\mathbf{v}_h^k$ suitable approximations of the exact solution $[\varrho, \mathbf{u}]$. To this end, we first extend $[\varrho, \mathbf{u}]$ to be defined on the numerical domains Ω_h. This can be done preserving the bounds (4.2), (4.3). We report the following result, see [2, Lemma 2.1]:

Lemma 5.1. *The exact solution $[\varrho, \mathbf{u}]$ can be extended as $[r, \mathbf{v}]$ outside Ω in such a way that:*

- *the extended density r is bounded below away from zero in $[0, T] \times R^3$, the extended velocity field $\mathbf{v}$ has compact support in $[0, T] \times R^3$;*
- *the equation of continuity*

$$\partial_t r + \mathrm{div}_x(r\mathbf{v}) = 0 \ \textit{holds in} \ (0, T) \times R^3; \tag{5.3}$$

- *we have the following estimates*

$$r|_\Omega = \varrho, \ \mathbf{v}|_\Omega = \mathbf{u},$$

$$\|\mathbf{v}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{v}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t \nabla_x \mathbf{v}\|_{C([0,T];L^6(R^3;R^{3\times 3}))} + \|\partial_{t,t}^2 \mathbf{v}\|_{C([0,T];L^6(R^3;R^3))} \tag{5.4}$$
$$\lesssim \|\mathbf{u}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{u}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{u}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial_{t,t}^2 \mathbf{u}\|_{C([0,T];L^6(\Omega;R^3))}$$

$$\|1/r\|_{C([0,T]\times R^3)} + \|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla_x r\|_{C([0,T];L^6_{\mathrm{loc}}(R^3;R^3))} + \|\partial_{t,t}^2 r\|_{C([0,T];L^6_{\mathrm{loc}}(R^3))} \tag{5.5}$$
$$\lesssim \|1/\varrho\|_{C([0,T]\times\overline{\Omega})} + \|\varrho\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla_x \varrho\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial_{t,t}^2 \varrho\|_{C([0,T];L^6(\Omega))}$$
$$+ \|\mathbf{u}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{u}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{u}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial_{t,t}^2 \mathbf{u}\|_{C([0,T];L^6(\Omega;R^3))}.$$

5.3 Ansatz in the relative energy inequality

Following the strategy of [2], we take

$$r_h^k = \Pi_h^Q[r(k\Delta t, \cdot)],\ k = 0, 1, \dots n$$

as a "test" function in the relative energy inequality (5.2), where r is the extension of the exact solution ϱ and Π_h^Q is the projection onto the space of piece-wise constant functions introduced in (2.3).

Similarly, one is tempted to take $\mathbf{v}_h^k = \Pi_h^V[\mathbf{v}(k\Delta t, \cdot)]$, with Π_h^V given by (2.6). Unfortunately, this is not a legitimate test function as, in general, $\Pi_h^V[\mathbf{v}(k\Delta t, \cdot)] \notin V_{0,h}(\Omega_h; R^3)$. Instead, following [2], we introduce a projection

$$\Pi_{h,0}^V : W^{1,q}(\Omega_h) \to V_{h,0}(\Omega_h),\ \int_\Gamma \Pi_{h,0}^V[v]\ \mathrm{dS}_x = \int_\Gamma v\ \mathrm{dS}_x \text{ if } \Gamma \in \Gamma_{h,\mathrm{int}},\ \int_\Gamma \Pi_{h,0}^V[v]\ \mathrm{dS}_x = 0 \text{ if } \Gamma \in \Gamma_{h,\mathrm{ext}}. \tag{5.6}$$

We have, see [2, Lemma 2.3, Corollary 2.1]:

$$\left\| \Pi_h^V[\phi] - \Pi_{h,0}^V[\phi] \right\|_{L^\infty(E)} + h \left\| \nabla_x \Pi_h^V[\phi] - \nabla_x \Pi_{h,0}^V[\phi] \right\|_{L^\infty(E;R^3)} \lesssim \sup_{\Gamma\subset\partial E,\ \Gamma\in\Gamma_{h,\mathrm{ext}}} \|\phi\|_{L^\infty(\Gamma)} \tag{5.7}$$

for any $E \in E_h$, $\phi \in C(E)$, and

$$\left\| \Pi_h^V[\phi] - \Pi_{h,0}^V[\phi] \right\|_{L^\infty(E)} + h \left\| \nabla_x \Pi_h^V[\phi] - \nabla_x \Pi_{h,0}^V[\phi] \right\|_{L^\infty(E;R^3)} \lesssim h \|\nabla_x \phi\|_{L^\infty(R^3;R^3)} \tag{5.8}$$

for any $E \in E_h$, $\phi \in C^1(R^3)$, $\phi|_{\partial\Omega} = 0$.

Remark 5.1. *Note that (5.8) is worse than its counterpart in [2, Corollary 2.1] due to the rough domain approximation considered in the present paper.*

The desired error estimates (3.3) will be deduced from the relative energy inequality (5.2) evaluated for the test functions

$$r_h^k = \Pi_h^Q[r(k\Delta t, \cdot)],\ \mathbf{v}_h^k = \Pi_{h,0}^V[\mathbf{v}(k\Delta t, \cdot)] \text{ for } k = 0, 1, \dots n,$$

where r, $\mathbf{v}$ is the extension of the exact solution $[\varrho, \mathbf{u}]$ and $n \geq T/\Delta t$.

6 Error estimates

Our ultimate goal is to establish the error estimates claimed in (3.3).

6.1 Energy estimates

We start by recalling the energy estimates for the family of approximate solutions. They can be deduced easily taking $r_h^k = 1$, $\mathbf{v}_h^k = 0$ in the relative energy inequality (5.2):

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_h} \varrho_h |\widehat{\mathbf{u}}_h|^2(t, \cdot)\ \mathrm{d}x \lesssim 1, \tag{6.1}$$

$$\int_0^T \int_{\Omega_h} |\nabla_h \mathbf{u}_h|^2\ \mathrm{d}x\ \mathrm{dt} \lesssim 1, \tag{6.2}$$

and, by virtue of the discrete analogue of the Sobolev embedding,

$$\int_0^T \|\mathbf{u}_h\|^2_{L^6(\Omega_h;R^3)}\ \mathrm{dt} \lesssim 1. \tag{6.3}$$

6.2 Perturbations

The crucial observation is that the proof of (5.2) reduces to that of [2, Theorem 3.1] provided we can "replace"

$$\mathbf{v}_h^k = \Pi_{h,0}^V[\mathbf{v}(k\Delta t, \cdot)] \text{ by } \tilde{\mathbf{v}}_h^k = \Pi_h^V[\mathbf{v}(k\Delta t, \cdot)]$$

in the relative entropy inequality (5.2), therefore we revisit [2, Lemma 6.1.]. Consequently, we have to show that the remainders resulting from such a procedure remain small. We proceed in several steps handling term by term the integrals $\mathcal{R}_{h,j}^k$. In what follows, we denote $\tilde{\mathcal{R}}_{h,1}^k$ the expression $\mathcal{R}_{h,1}^k$ with $\mathbf{v}_1^k$ replaced by $\tilde{\mathbf{v}}_1^k$.

6.2.1 Remainder term $\mathcal{R}_{h,1}^k$

We have to control

$$\int_{\Omega_h} \nabla_h \left(\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k\right) : \nabla_h \left(\mathbf{u}_h^k - \mathbf{v}_h^k\right) \ \mathrm{d}x \text{ and } \int_{\Omega_h} \nabla_h \left(\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k\right) : \nabla_h \tilde{\mathbf{v}}_h^k \ \mathrm{d}x. \tag{6.4}$$

Since $\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k = 0$ in E whenever $E \cap \partial\Omega_h = \emptyset$, we have

$$\int_{\Omega_h} \nabla_h \left(\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k\right) : \left[\nabla_h \left(\mathbf{u}_h^k - \mathbf{v}_h^k\right) + \nabla_h \tilde{\mathbf{v}}_h^k\right] \ \mathrm{d}x = \int_{U_h} \nabla_h \left(\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k\right) : \left[\nabla_h \left(\mathbf{u}_h^k - \mathbf{v}_h^k\right) + \nabla_h \tilde{\mathbf{v}}_h^k\right] \mathrm{d}x,$$

where

$$|U_h| \stackrel{<}{\sim} h. \tag{6.5}$$

By virtue of (5.8),

$$\left\|\nabla_h \left(\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k\right)\right\|_{L^\infty(U_h;R^3)} \stackrel{<}{\sim} 1,$$

and, in accordance with (6.2),

$$\sup_{t\in(0,T)} \left\|\nabla_h \tilde{\mathbf{v}}_h^k(t,\cdot)\right\|_{L^\infty(U_h;R^3)} + \int_0^T \left\|\nabla_h \left(\mathbf{v}_h^k - \mathbf{u}_h^k\right)\right\|_{L^2(U_h;R^3)}^2 \ \mathrm{d}t \stackrel{<}{\sim} 1.$$

Seeing that the integral containing div_h can be treated in the same manner, we may infer that

$$\Delta t \sum_{k=1}^n \left|\mathcal{R}_{h,1}^k - \tilde{\mathcal{R}}_{h,1}^k\right| \stackrel{<}{\sim} h^{1/2}. \tag{6.6}$$

6.2.2 Remainder term $\mathcal{R}_{h,2}^k$

Obviously, the most difficult term to handle reads

$$\int_{\Omega_h} \varrho_h^{k-1} \mathbf{u}_h^{k-1} \cdot \left(\frac{\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k}{\Delta t}\right) \ \mathrm{d}x.$$

Again, we reduce the integration domain from Ω_h to U_h. Then, in view of (5.8) and the hypothesis $\Delta t \approx h$, we have

$$\left|\int_{U_h} \varrho_h^{k-1} \mathbf{u}_h^{k-1} \cdot \left(\frac{\mathbf{v}_h^k - \tilde{\mathbf{v}}_h^k}{\Delta t}\right) \mathrm{d}x\right| \stackrel{<}{\sim} r \left|\int_{U_h} |\mathbf{u}_h^{k-1}| \ \mathrm{d}x\right| \stackrel{<}{\sim} h^{1/2} \|\mathbf{u}_h^{k-1}\|_{L^2(\Omega_h;R^3)}.$$

Seeing that the other integrals in $\mathcal{R}_{h,2}^k$ may be treated in a similar way, we may use the energy bound (6.3) to conclude

$$\Delta t \sum_{k=1}^n \left|\mathcal{R}_{h,2}^k - \tilde{\mathcal{R}}_{h,2}^k\right| \stackrel{<}{\sim} h^{1/2}. \tag{6.7}$$

6.2.3 Remainder term $\mathcal{R}^k_{h,3}$

The most difficult term is of the type

$$\sum_{\Gamma\in\Gamma_{h,\rm int}} \int_\Gamma \varrho_h^k |\widehat{\mathbf{u}}_h^k| \ \left| \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \ \left| \left[\left[\widehat{\mathbf{v}}_h^k - \widehat{\widehat{\mathbf{v}}}{}_h^k \right]\right] \right| \ {\rm dS}_x,$$

where, by virtue of (5.8),

$$\left| \left[\left[\widehat{\mathbf{v}}_h^k - \widehat{\widehat{\mathbf{v}}}{}_h^k \right]\right] \right| \lesssim h,$$

and the sum is taken only over $\Gamma \subset E$, such that $E \subset U_h$, where the set $U_h \subset \Omega_h$ satisfies (6.5). Consequently, we get

$$\left| \sum_{\Gamma\in\Gamma_{h,\rm int}} \int_\Gamma \varrho_h^k |\widehat{\mathbf{u}}_h^k| \ \left| \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \ \left| \left[\left[\widehat{\mathbf{v}}_h^k - \widehat{\widehat{\mathbf{v}}}{}_h^k \right]\right] \right| \ {\rm dS}_x \right| \lesssim h \sum_{\Gamma\in\Gamma_{h,\rm int}, \Gamma\subset\partial E, E\subset U_h} \int_\Gamma |\widehat{\mathbf{u}}_h^k| \ \left| \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \ {\rm dS}_x$$

$$\lesssim \int_{U_h} |\widehat{\mathbf{u}}_h^k| \ |\mathbf{u}_h^k| \ {\rm d}x.$$

Finally, in accordance with (6.3),

$$\widehat{\mathbf{u}}_h, \mathbf{u}_h \in L^2(0,T; L^6(\Omega_h; R^3)),$$

and we may infer, exactly as in the previous step,

$$\Delta t \sum_{k=1}^n \left| \mathcal{R}^k_{h,3} - \tilde{\mathcal{R}}^k_{h,3} \right| \lesssim h^{2/3}. \tag{6.8}$$

6.2.4 Remainder term $\mathcal{R}^k_{h,4}$

In view of (5.8) and hypothesis (3.2), we get

$$\left| \mathcal{R}^k_{h,4} - \tilde{\mathcal{R}}^k_{h,4} \right| \lesssim \int_{U_h} 1 \ {\rm d}x,$$

where U_h is the same as in (6.5); whence

$$\Delta t \sum_{k=1}^n \left| \mathcal{R}^k_{h,4} - \tilde{\mathcal{R}}^k_{h,4} \right| \lesssim h. \tag{6.9}$$

As observed at the beginning of this section, the estimates (6.6–6.9) reduce the proof of Theorem 3.1 to the arguments used in the proof [2, Theorem 3.1].

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